1. This question refers to the $k$-Median problem.
   (a) Prove the following claims, which are part of the analysis of the 5-approximation algorithm given in class. Recall that we are looking at one critical swap, in which $i^* \in S^*$ and $i \in S$ are the swapped facilities. Let $S'$ be the set of facilities after this swap, i.e., $S' = (S \setminus \{i\}) \cup \{i^*\}$.
   i. The new assignment is valid, i.e., every customer $j$ is assigned to a facility in $S'$.
   ii. The cost of $S'$ is at least the cost of $S$.
(b) Next, we want to generalize the algorithm given in class to use multiswaps, in which up to $p \geq 1$ facilities can be swapped simultaneously. The neighborhood structure is now defined by $S' = \{(S \setminus A) \cup B : A \subseteq S, B \subseteq N, |A| = |B| \leq p\}$. This definition captures the set of solutions obtainable by deleting a set of at most $p$ facilities, $A$, and adding a set of facilities, $B$, of the same size. Prove that the locality gap of the $k$-Median problem with respect to this operation is $(3 + \frac{2}{p})$.

2. Let $G = (V, E)$ be an undirected graph. In the Max Leaf Spanning Tree problem, the goal is to find a spanning tree of $G$ that has a maximum number of leaves. Suggest a 10-approximation algorithm for this problem, using the local search technique.

3. Recall that, in the Minimum Weight Set Cover (MWSC) problem, we have a universe $U = \{u_1, u_2, \ldots, u_n\}$, a set of subsets $S = \{S_1, S_2, \ldots, S_m\} \in 2^U$, and a weight function $w : S \to \mathbb{N}$. The goal is to find a subset of indices $I \subseteq \{1, 2, \ldots, m\}$ such that $\cup_{j \in I} S_j = U$, and $w(I) \triangleq \sum_{j \in I} w(S_j)$ is minimized. Let $f$ be the maximal frequency of an item, i.e., $f \triangleq \max_{u \in U} |\{S_j \in S \mid u \in S_j\}|$. Suggest an $f$-approximation algorithm for the MWSC problem.

4. Let $G = (V, E)$ be an undirected graph, and let $w : V \to \mathbb{Q}^+$ be a vertex weight function. In the Minimum Weight Feedback Vertex Set (MWFVS) problem, the goal is to find a set of vertices $F \subseteq V$ of minimal total weight, such that every cycle in $G$ contains some vertex from $F$. A semi-disjoint cycle $C$ is a cycle where every vertex $v \in C$ has degree of 2, with at most one exception (i.e., the cycle may be connected to other parts in the graph through at most a single vertex). Figure 1 gives a local ratio algorithm for the problem due to Bafna et al. [1]. The paper shows that the algorithm outputs a 2-approximation for MWFVS, and that the analysis is tight. Consider the case of uniform weights (i.e., $\forall v \in V : w(v) = 1$). Show that the algorithm does not achieve approximation ratio better than $2 - o(1)$ even for this case.
Input: an undirected graph $G = (V, E)$ with vertex weights $w : V \rightarrow \mathbb{Q}_+$
Output: a feedback vertex set $F$

Initialize $F = \{u \in V : w(u) = 0\}, V = V - F$. [$i = 0$]

Cleanup($G$)
While $V \neq \emptyset$ do

[$i \leftarrow i + 1$]
If $G$ contains a semidisjoint cycle $C$, then
Let $\gamma \leftarrow \min\{w(u) : u \in V(C)\}$.
Set $w(u) \leftarrow w(u) - \gamma$, $\forall u \in V(C)$.
$[G_i = C$ and $w_i(u) = \gamma$, $\forall u \in V(C)]$
Else [$G$ is clean and contains no semidisjoint cycle]
Let $\gamma \leftarrow \min\{w(u)/(d(u) - 1) : u \in V\}$.
Set $w(u) \leftarrow w(u) - \gamma(d(u) - 1)$, $\forall u \in V$.
$[G_i = G$ and $w_i(u) = \gamma(d(u) - 1)$, $\forall u \in V]$
For each $u \in V$ with $w(u) = 0$ do
Remove $u$ from $V$, add it to $F$, and push it onto STACK.

Cleanup($G$)
While STACK $\neq \emptyset$ do

Let $u \leftarrow \text{pop}(\text{STACK})$.
If $F - \{u\}$ is an FVS in original $G$, then $[u$ is redundant$]$
Remove $u$ from $F$.

Cleanup($G$):
While $G$ contains a vertex of degree at most 1, remove it along with any incident edges.

Figure 1: The local ratio algorithm for MWFVS by Bafna et al. [1].

References