

Introduction to Number Theory 2

Quadratic Residues

Definition: The numbers $0^2, 1^2, 2^2, \dots, (n-1)^2 \pmod n$, are called **quadratic residues** modulo n . Numbers which are not quadratic residues modulo n are called **quadratic non-residues** modulo n .

Example: Modulo 11:

i	0	1	2	3	4	5	6	7	8	9	10
$i^2 \pmod{11}$	0	1	4	9	5	3	3	5	9	4	1

There are six quadratic residues modulo 11: 0, 1, 3, 4, 5, and 9.

There are five quadratic non-residues modulo 11: 2, 6, 7, 8, 10.

Quadratic Residues (cont.)

Lemma: Let p be prime. Exactly half of the numbers in Z_p^* are quadratic residues. With 0, exactly $\frac{p+1}{2}$ numbers in Z_p are quadratic residues.

Proof: There are at most $\frac{p+1}{2}$ quadratic residues, since

$$\begin{aligned} &0^2 \\ &1^2 \equiv (p-1)^2 \pmod p \\ &2^2 \equiv (p-2)^2 \pmod p \\ &\vdots \\ &i^2 \equiv (p-i)^2 \pmod p \quad \forall i \\ &\vdots \end{aligned}$$

Thus, all the elements in Z_p span at most $\frac{p+1}{2}$ quadratic residues.

There are at least $\frac{p+1}{2}$ quadratic residues, otherwise, for some $i \neq j \leq \frac{p-1}{2}$ it holds that $i^2 = (p-i)^2 = j^2 = (p-j)^2$, in contrast to Lagrange theorem that states that the equation $x^2 - i^2 = 0 \pmod p$ has at most two solutions.

Quadratic Residues (cont.)

Since Z_p^* is cyclic, there is a generator. Let g be a generator of Z_p^* .

1. g is a quadratic non-residue modulo p , since otherwise there is some b such that $b^2 \equiv g \pmod p$. Clearly, $b^{p-1} \equiv 1 \pmod p$, and thus $g^{\frac{p-1}{2}} \equiv b^{p-1} \equiv 1 \pmod p$. However, the order of g is $p-1$. Contradiction.
2. $g^2, g^4, \dots, g^{(p-1)}$ mod p are quadratic residues, and are distinct, therefore, there are at least $\frac{p-1}{2}$ quadratic residues.
3. $g, g^3, g^5, \dots, g^{(p-2)}$ mod p are quadratic non-residues, since if any of them is a quadratic residue, g is also a quadratic residue.

QED

Euler's Criterion

Theorem: Let $p \neq 2$ be a prime, and let $a \in Z_p^*$. Then, a is a quadratic residue modulo p iff $a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$.

Proof:

(\Rightarrow) If a is a quadratic residue, there is some b such that $a \equiv b^2 \pmod{p}$. Thus,

$$a^{\frac{p-1}{2}} \equiv (b^2)^{\frac{p-1}{2}} \equiv b^{p-1} \equiv 1 \pmod{p}.$$

Euler's Criterion (cont.)

(\Leftarrow) If a is a quadratic non-residue: For any r there is a unique s such that $rs \equiv a \pmod{p}$, i.e., $s = ar^{-1}$, and there is no $r^* \neq r$ such that $s = ar^{*-1}$. Since a is a quadratic non-residue, $r \not\equiv s \pmod{p}$.

Thus, the numbers $1, 2, 3, \dots, p-1$ are divided into $\frac{p-1}{2}$ distinct pairs $(r_1, s_1), (r_2, s_2), \dots, (r_{\frac{p-1}{2}}, s_{\frac{p-1}{2}})$, such that $r_i s_i = a$, and we get

$$\begin{aligned} a^{\frac{p-1}{2}} &\equiv r_1 s_1 r_2 s_2 \dots r_{\frac{p-1}{2}} s_{\frac{p-1}{2}} \equiv \\ &\equiv 1 \cdot 2 \cdot \dots \cdot (p-1) \equiv -1 \pmod{p} \end{aligned}$$

by Wilson's theorem. QED

Quadratic Residues Modulo $n = pq$

Let p and q be large primes and let $n = pq$ (as in RSA).

Theorem: Let $m \in Z_n^*$. If m is a quadratic residue modulo n , then m has exactly **four** square roots modulo n in Z_n^* .

Proof: Assume $\alpha^2 \equiv m \pmod{n}$. Then

$$\gcd(m, n) = 1 \Rightarrow \gcd(\alpha^2, n) = 1 \Rightarrow \gcd(\alpha, n) = 1 \Rightarrow \alpha \in Z_n^*.$$

and since

$$m \equiv \alpha^2 \pmod{n}$$

then

$$\begin{aligned} m &\equiv \alpha^2 \pmod{p} \\ m &\equiv \alpha^2 \pmod{q} \end{aligned}$$

m has two square roots modulo p ($\alpha \pmod{p}$ and $-\alpha \pmod{p}$) and two square roots modulo q ($\alpha \pmod{q}$ and $-\alpha \pmod{q}$).

Quadratic Residues Modulo $n = pq$ (cont.)

Look at the systems of equations

$$\begin{aligned} x &\equiv \pm\alpha \pmod{p} \\ x &\equiv \pm\alpha \pmod{q} \end{aligned}$$

which represent four systems (one of each possible choice of \pm). Each system has an unique solution modulo n which satisfies

$$\begin{aligned} x^2 &\equiv m \pmod{p} \\ x^2 &\equiv m \pmod{q} \end{aligned}$$

and thus satisfies

$$x^2 \equiv m \pmod{n}$$

All the four solutions are roots of m modulo n .

These are all the roots. Otherwise there must be more than two roots either modulo p or modulo q .

QED

Quadratic Residues Modulo $n = pq$ (cont.)

Conclusion: Exactly a quarter of the numbers in Z_n^* are quadratic residues modulo n .

Legendre's Symbol

Definition: Let p be a prime such that $p \nmid a$. **Legendre's symbol** of a over p is

$$\left(\frac{a}{p}\right) \triangleq \begin{cases} +1, & \text{if } a \text{ is a quadratic residue modulo } p; \\ -1, & \text{if } a \text{ is a quadratic non-residue modulo } p. \end{cases}$$

By Euler:

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}.$$

Legendre's Symbol (cont.)

Properties of Legendre's symbol:

1. $a \equiv a' \pmod{p} \Rightarrow \left(\frac{a}{p}\right) = \left(\frac{a'}{p}\right)$.
2. $\left(\frac{1}{p}\right) = \left(\frac{c^2}{p}\right) = 1 \quad \forall c$.
3. $\left(\frac{-1}{p}\right) = \begin{cases} 1, & \text{if } p = 4k + 1; \\ -1, & \text{if } p = 4k + 3. \end{cases}$

Proof:

$$\begin{aligned} \left(\frac{-1}{p}\right) &\equiv (-1)^{\frac{p-1}{2}} \pmod{p} \\ &\equiv \begin{cases} (-1)^{\frac{4k+1-1}{2}} \equiv (-1)^{2k} \equiv 1, & \text{if } p = 4k + 1; \\ (-1)^{\frac{4k+3-1}{2}} \equiv (-1)^{2k+1} \equiv -1, & \text{if } p = 4k + 3. \end{cases} \end{aligned}$$

Legendre's Symbol (cont.)

4. $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$.
(given without a proof).
5. $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right) \left(\frac{b}{p}\right)$.

Proof:

Let g be a generator modulo p . Then, $\exists i, a \equiv g^i \pmod{p}$ and $\exists j, b \equiv g^j \pmod{p}$. a is a quadratic residue iff i is even, b is a quadratic residue iff j is even, and ab is a quadratic residue iff $i + j$ is even. Thus, by Euler:

$$\left(\frac{ab}{p}\right) \equiv (-1)^{i+j} \equiv (-1)^i (-1)^j \equiv \left(\frac{a}{p}\right) \left(\frac{b}{p}\right) \pmod{p}.$$

Legendre's Symbol (cont.)

6. The reciprocity law: if $p \neq q$ are both odd primes then

$$\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2} \frac{q-1}{2}} \left(\frac{q}{p}\right).$$

(given without a proof).

Jacobi's Symbol

Jacobi's symbol is a generalization of Legendre's symbol to composite numbers.

Definition: Let n be odd, and let p_1, p_2, \dots, p_k be the prime factors of n (not necessarily distinct) such that $n = p_1 p_2 \cdots p_k$. Let a be coprime to n .

Jacobi's symbol of a over n is

$$\left(\frac{a}{n}\right) \triangleq \left(\frac{a}{p_1}\right) \left(\frac{a}{p_2}\right) \cdots \left(\frac{a}{p_k}\right).$$

In particular, for $n = pq$

$$\left(\frac{a}{n}\right) = \left(\frac{a}{pq}\right) = \left(\frac{a}{p}\right) \left(\frac{a}{q}\right).$$

Jacobi's Symbol (cont.)

Remarks:

1. $a \in Z_n^*$ is a quadratic residue modulo n iff the Legendre's symbols over all the prime factors are 1.
2. When Jacobi's symbol is 1, a is not necessarily a quadratic residue.
3. When Jacobi's symbol is -1, a is necessarily a quadratic non-residue.

Jacobi's Symbol (cont.)

Properties of Jacobi's symbol:

Let m and n be integers, and let a and b be coprime to m and n . Assume that n is odd and that the factorization of n is $n = p_1 p_2 \cdots p_k$.

1. $a \equiv b \pmod{n} \Rightarrow \left(\frac{a}{n}\right) = \left(\frac{b}{n}\right)$.
2. $\left(\frac{1}{n}\right) = 1 \quad \forall n$ (1 is a quadratic residue modulo any n).
3. $\left(\frac{-1}{n}\right) = (-1)^{\frac{n-1}{2}}$.

Proof:

$$\begin{aligned} n &= p_1 p_2 \cdots p_k \\ &= ((p_1 - 1) + 1)((p_2 - 1) + 1) \cdots ((p_k - 1) + 1) \end{aligned}$$

opening parentheses:

$$= \sum_{S \subseteq \{1, 2, \dots, k\}} \prod_{i \in S} (p_i - 1)$$

Jacobi's Symbol (cont.)

$$= \left[\sum_{\substack{S \subseteq \{1,2,\dots,k\} \\ |S| \geq 2}} \prod_{i \in S} (p_i - 1) \right] + \sum_{i \in \{1,2,\dots,k\}} (p_i - 1) + 1$$

$$= [(p_1 - 1)(p_2 - 1) \cdots (p_k - 1) + \dots] + (p_1 - 1) + (p_2 - 1) + \dots + (p_k - 1) + 1$$

where all the terms with $|S| \geq 2$ (in the brackets) are multiples of four, and all the $p_i - 1$ are even. Thus,

$$\frac{n-1}{2} \equiv \frac{p_1-1}{2} + \frac{p_2-1}{2} + \dots + \frac{p_k-1}{2} \pmod{2},$$

and

$$\left(\frac{-1}{n}\right) = \left(\frac{-1}{p_1}\right) \left(\frac{-1}{p_2}\right) \cdots \left(\frac{-1}{p_k}\right)$$

$$= (-1)^{(p_1-1)/2} (-1)^{(p_2-1)/2} \cdots (-1)^{(p_k-1)/2}$$

$$= (-1)^{(p_1-1)/2 + (p_2-1)/2 + \dots + (p_k-1)/2} = (-1)^{(n-1)/2}.$$

Jacobi's Symbol (cont.)

$$4. \left(\frac{2}{n}\right) = (-1)^{\frac{n^2-1}{8}}.$$

Proof:

We saw that $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$, thus:

$$\left(\frac{2}{n}\right) = \left(\frac{2}{p_1}\right) \left(\frac{2}{p_2}\right) \cdots \left(\frac{2}{p_k}\right) = (-1)^{\frac{p_1^2-1}{8} + \frac{p_2^2-1}{8} + \dots + \frac{p_k^2-1}{8}}$$

It remains to show that

$$\frac{n^2-1}{8} \equiv \frac{p_1^2-1}{8} + \frac{p_2^2-1}{8} + \dots + \frac{p_k^2-1}{8} \pmod{2}$$

$$q_1^2 q_2^2 = (1 + (q_1^2 - 1))(1 + (q_2^2 - 1))$$

$$= 1 + (q_1^2 - 1) + (q_2^2 - 1) + (q_1^2 - 1)(q_2^2 - 1)$$

But $8|(q_1^2 - 1)$ and $8|(q_2^2 - 1)$, thus $64|(q_1^2 - 1)(q_2^2 - 1)$. Therefore,

$$q_1^2 q_2^2 \equiv 1 + (q_1^2 - 1) + (q_2^2 - 1) \pmod{16}$$

Jacobi's Symbol (cont.)

And,

$$q_1^2 q_2^2 q_3^2 \equiv (1 + (q_1^2 - 1))(1 + (q_2^2 - 1))(1 + (q_3^2 - 1)) \pmod{16}$$

$$\equiv 1 + (q_1^2 - 1) + (q_2^2 - 1) + (q_3^2 - 1) \pmod{16}$$

etc., thus,

$$n^2 \equiv 1 + (q_1^2 - 1) + (q_2^2 - 1) + \dots + (q_k^2 - 1) \pmod{16}$$

$$\frac{n^2-1}{8} \equiv \frac{p_1^2-1}{8} + \frac{p_2^2-1}{8} + \dots + \frac{p_k^2-1}{8} \pmod{2}$$

Jacobi's Symbol (cont.)

$$5. \text{ The first multiplication property: } \left(\frac{a}{mn}\right) = \left(\frac{a}{m}\right) \left(\frac{a}{n}\right).$$

(if a is coprime to mn it is coprime to m and to n ; the rest is derived directly from the definition).

$$6. \text{ The second multiplication property: } \left(\frac{ab}{n}\right) = \left(\frac{a}{n}\right) \left(\frac{b}{n}\right).$$

(if ab is coprime to n , then both a and b are coprime to n ; the rest is derived since this property holds for Legendre's symbol).

Jacobi's Symbol (cont.)

7. The reciprocity law: if m, n are coprime and odd then

$$\left(\frac{n}{m}\right) = (-1)^{\frac{m-1}{2} \cdot \frac{n-1}{2}} \left(\frac{m}{n}\right).$$

Proof:

First assume that $m = q$ is a prime, thus,

$$\left(\frac{n}{q}\right) = \left(\frac{p_1}{q}\right) \left(\frac{p_2}{q}\right) \cdots \left(\frac{p_k}{q}\right).$$

By the reciprocity law of Legendre's symbol we know that

$$\left(\frac{p_i}{q}\right) = (-1)^{\frac{p_i-1}{2} \cdot \frac{q-1}{2}} \left(\frac{q}{p_i}\right).$$

Thus,

$$\left(\frac{n}{q}\right) = (-1)^{\frac{q-1}{2}(\frac{p_1-1}{2} + \dots + \frac{p_k-1}{2})} \underbrace{\left(\frac{q}{p_1}\right) \left(\frac{q}{p_2}\right) \cdots \left(\frac{q}{p_k}\right)}_{\left(\frac{q}{n}\right)}.$$

Jacobi's Symbol (cont.)

We saw in property 3 that,

$$\frac{n-1}{2} \equiv \frac{(p_1-1)}{2} + \frac{(p_2-1)}{2} + \dots + \frac{(p_k-1)}{2} \pmod{2},$$

thus,

$$\left(\frac{n}{q}\right) = (-1)^{\frac{q-1}{2} \cdot \frac{n-1}{2}} \left(\frac{q}{n}\right).$$

Now for any odd m :

$$\begin{aligned} \left(\frac{n}{m}\right) &= \left(\frac{n}{q_1}\right) \left(\frac{n}{q_2}\right) \cdots \left(\frac{n}{q_\ell}\right) \\ &= \left(\frac{q_1}{n}\right) \left(\frac{q_2}{n}\right) \cdots \left(\frac{q_\ell}{n}\right) (-1)^{\frac{n-1}{2}(\frac{q_1-1}{2} + \dots + \frac{q_\ell-1}{2})} \\ &= (-1)^{\frac{m-1}{2} \cdot \frac{n-1}{2}} \left(\frac{m}{n}\right) \end{aligned}$$

QED

Jacobi's Symbol (cont.)

Application of Jacobi's Symbol:

Using the properties of Jacobi's symbol, it is easy to calculate Legendre's symbols in polynomial time.

Example:

$$\begin{aligned} \left(\frac{117}{271}\right) &= +1 \cdot \left(\frac{271}{117}\right) = \left(\frac{37}{117}\right) = \left(\frac{117}{37}\right) = \left(\frac{6}{37}\right) \\ &= \left(\frac{2}{37}\right) \left(\frac{3}{37}\right) = (-1) \left(\frac{3}{37}\right) = (-1)(+1) \left(\frac{37}{3}\right) \\ &= (-1)(+1) \left(\frac{1}{3}\right) = (-1)(+1)1 = -1 \end{aligned}$$

271 is prime, therefore $\left(\frac{117}{271}\right)$ can also be computed by:

$$\left(\frac{117}{271}\right) \equiv 117^{\frac{271-1}{2}} \equiv 117^{135} \equiv -1 \pmod{271}.$$

Jacobi's Symbol (cont.)

Complexity:

The only required arithmetic operations are modular reductions and division by powers of two.

Clearly, a division (rule 6) reduces the "numerator" by a factor of two. A modular reduction (using rule 7 and then rule 1), reduces the number by at least two: as if $a > b$ then $a = qb + r \geq b + r > r + r$, thus $r < a/2$, i.e., $a \bmod b < a/2$.

Therefore, at most $O(\log n)$ modular reductions/divisions are performed, each of which takes $O((\log n)^2)$ time. This shows that the complexity is $O((\log n)^3)$, which is polynomial in $\log n$.

A more precise analysis of this algorithm shows that the complexity can be reduced to $O((\log n)^2)$.