Theorem: For every algorithm $A$ for maximum finding in unidirectional rings, and every set $I$ of $n$ identities,

$$\text{avg}_A(I) \geq \Omega(n \log n)$$
Sequence $s = (s_1, s_2, \ldots, s_t)$

Prefix of a sequence $s' = (s_1, s_2, \ldots, s_r)$, $1 \leq r \leq t$

Concatenation of sequences $s \cdot u$

subsequence of $s$ if $\exists r, t$ s.t. $s = ru\cdot t$

$\mathcal{C}(S) =$ all cyclic permutations of $S$

$|\mathcal{C}(s)| = \text{length} (s)$
In every execution of a maximum finding algorithm A, at least one processor must see its own value.
In a ring labeled \( s \), at least one message in \( C(s) \) is sent by A.

\[
\begin{align*}
\text{In every execution of a maximum finding algorithm A, at least one processor must see its own value.}
\end{align*}
\]
Note: though we wrote $s \in D, E \subseteq D, |E| < \infty$:

E does not have to be a finite set.

Why?

$s = (4, 1, 3, 5)$
$E = \{(4, 5), (3, 5), (5, 4), (6, 3), (5, 4)\}$

$N_0(s, E) = 4$
$N_1(s, E) = 1$
$N_2(s, E) = 2$
$N_3(s, E) = 1$
$N_k(s, E) = 0$ for $k \geq 4$

Definition: A set $E \subseteq D$ is exhaustive if it has the following two properties:

- Prefix property:
  - If $u \in E$ then $s \in E$ for every prefix $s$ of $u$.
- Cyclic permutation property:
  - $\forall s \in D: C(s) \cap E \neq \emptyset$
Example: the set \(E = \{(s_1, s_2, ..., s_k) \mid s_1 = \max\{s_1, s_2, ..., s_k\}\}\)

E contains also the following:
1. (4), (4,1), (4,1,3),
2. (1),
3. (3),
4. (5), (5,1), (5,1,3), (5,1,3)

E is the set of messages sent by the Chang & Roberts’ algorithm!

Lemma: Let \(s, t, u \in D\) such that \(u\) is a prefix of \(s\) and \(t\), and let \(A\) be a maximum finding algorithm.

If in the execution of \(A\) on ring \(s\) a message \(u\) is sent, then in the execution of \(A\) on ring \(t\) a message \(u\) is sent.
**Theorem:** For every maximum finding algorithm $A$ for unidirectional rings, there exists an exhaustive set $E(A)$, such that, for every ring $s$, $A$ sends at least $N(s,E(A))$ messages on $s$.

**Proof:** Let

$$E(A) = \{ s \in D \mid \text{a message } s \text{ is sent when } A \text{ executes on ring } s \}$$

1. $E(A)$ is exhaustive
   1a. Prefix property
   1b. Cyclic permutation property

   for a ring $s$, at least one processor must send a message $t \in C(s)$
   $t \in E(A)$
2. At least $N(s, E(A))$ messages sent by $A$ on $s$ 
$f \in E(A)$ 
$f$ is a subsequence of $s$ 
message $f$ was sent on ring $f$ 
b by Lemma: message $f$ was sent on ring $s$ 
at least $N(s, E(A))$ messages were sent on $s$

Theorem: For every maximum finding algorithm $A$ for unidirectional rings, and a set of $n$ identities $I$, we have:

$$\text{ave}_A(I) \geq \frac{1}{n!} \sum_{s \in \text{perm}(I)} N(s, E(A))$$

$$\text{worst}_A(I) \geq \max_{s \in \text{perm}(I)} N(s, E(A))$$

Theorem: For every maximum finding algorithm $A$ for unidirectional rings, and a set of $n$ identities $I$, we have:

$$\text{ave}_A(I) \geq \Omega(n \log n)$$
Theorem: In a unidirectional ring whose size $n$ is unknown, the Chang & Roberts algorithm has an optimal message complexity of

$$n(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}) \approx 0.69n \log n$$

Exercise: What if $n$ is known? What if synchronous?
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<thead>
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<th>References</th>
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<tbody>
<tr>
<td><strong>J. E. Burns</strong>,</td>
</tr>
<tr>
<td><em>A formal model for message passing systems</em>,</td>
</tr>
<tr>
<td>TR-91, Indiana University, September 1980.</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td><strong>E. Chang and R. Roberts</strong>,</td>
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<tr>
<td><strong>J. Pachl, E. Korach and D. Rotem</strong></td>
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</tbody>
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