Shape Matching & Correspondence

Based on slides by Maks Ovsjanikov
Overall Goal

Given two shapes, find **correspondences** between them.
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Given two shapes, find **correspondences** between them.

Finding the **best** map between a pair of shapes.
Applications

Manufacturing:
One shape is a **model** and the other is a **scan** of a product. Finding defects.

Medicine:
Finding correspondences between 3D MRI scans of the same person or different people.

Animation Reconstruction & 3D Video.

Statistical Shape Analysis:
Building models for a collection of shapes.
Applications – Statistical Analysis

Scan many people. Learn the deformation model (e.g. PCA)
Find the principal variation
create new random instances.
Requires correspondences.

A Statistical Model of Human Pose and Body Shape, Hasler et al. Eurographics 09
## Method Taxonomy

<table>
<thead>
<tr>
<th>Local vs. Global</th>
<th>Rigid vs. Deformable</th>
<th>Pair vs. Collection</th>
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<td>refinement (e.g. ICP) \ alignment (search)</td>
<td>rotation, translation \ general deformation</td>
<td>two shapes \ multiple shapes</td>
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Method Taxonomy

Local vs. Global
refinement (e.g. ICP) alignment (search)

Solved
Rigid vs. Deformable
rotation, translation general deformation

Pair vs. Collection
two shapes multiple shapes
Method Taxonomy

Local vs. Global refinement (e.g. ICP) | alignment (search)

Rigid vs. Deformable rotation, translation | general deformation

Pair vs. Collection two shapes | multiple shapes

Most other combinations are open
Pairwise Rigid Correspondence

Remember ICP:

Given a pair of shapes, $X$ and $Y$, iterate:

1. For each $x_i \in X$ find nearest neighbor $y_i \in Y$.
2. Find deformation $R, t$ minimizing:

$$\sum_{x_i \in X} d^2(Rx_i + t, y_i)$$
Pairwise Rigid Correspondence

Geometry of the square distance function

For a curve $\Psi$, around point $x$.

To second order:

$$d^2(y, \Psi) \approx \frac{d}{d - \rho_1} x_1^2 + x_2^2 \quad \text{in the Frenet frame at } p$$

[ Pottmann and Hofer 2003]
Approximate Squared Distance

For a curve $\Psi$, to second order:

$$d^2(y, \Psi) \approx \frac{d}{d - \rho_1} x_1^2 + x_2^2$$

For a surface $\Phi$, to second order:

$$d^2(y, \Phi) \approx \frac{d}{d - \rho_1} x_1^2 + \frac{d}{d - \rho_2} x_2^2 + x_3^2$$

$\rho_1 = 1/\kappa_1$ and $\rho_2 = 1/\kappa_2$ are inverse principal curvatures

[ Pottmann and Hofer 2003 ]
Approximate Squared Distance

For a surface $\Phi$, to second order:

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$\rho_1 = 1/\kappa_1$ and $\rho_2 = 1/\kappa_2$ are inverse principal curvatures

Note that as $d \to 0$, $d^2(y, \Phi) \to x_3^2$ point-to-plane

$\quad d \to \infty$, $d^2(y, \Phi) \to x_1^2 + x_2^2 + x_3^2$ point-to-point

In general neither metric guarantees second order consistency
\( d^2(y, \Phi_p) \) Using d2Tree

Partition the space into cells where each cell stores a quadratic approximant of the squared distance function.

LEOPOLDSEDER S et al. *d2-tree: A hierarchical representation of the squared distance function*
Registration using d2Tree

Build using bottom-up approach: fit a quadratic approximation to a fine grid.

Merge cells if they have similar approximations.

Funnel of Convergence:

Translation in x-z plane.
Rotation about y-axis.

Converges

Does not converge
Matching the Bunny to itself

Registration of Point Cloud Data from a Geometric Optimization Perspective
Mitra et al., SGP 2004
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Mitra et al., SGP 2004
Local Rigid Matching – ICP

The upshot is that

- Locally, the point-to-plane metric provides a second order approximation to the squared distance function.

- Optimization based on point-to-plane will converge quadratically to the local minimum.

- Convergence funnel can be narrow, but can improve it with either d2tree or point-to-point.

What if we are outside the convergence funnel?
Global Matching

Given shapes in *arbitrary* positions, find their alignment:

Robust Global Registration
Gelfand et al. SGP 2005

Can be approximate, since will refine later using e.g. ICP
Global Matching – Approaches

Several classes of approaches:

1. Exhaustive Search
2. Normalization
3. Random Sampling
4. Invariance
Exhaustive Search:

Compare at all alignments

- Sample the space of possible initial alignments.
- Correspondence is determined by the alignment at which models are closest.

Very common in biology: e.g., protein docking
Exhaustive Search:

Compare at all alignments

- Sample the space of possible initial alignments.
- Correspondence is determined by the alignment at which models are closest.

- Provides optimal result
- Can be unnecessarily slow.
- Does not generalize to non-rigid deformations.
Normalization

There are only a handful of initial configurations that are important.

Can center all shapes at the origin and use PCA to find the principal directions of the shape.

In addition sometimes try to all permutations of x-y-z.
Normalization

There are only a handful of initial configurations that are important.

Works well if we have complete shapes and no noise.

Fails for partial scans, outliers, high noise, etc.
Random Sampling (RANSAC)

ICP only needs 3 point pairs!

Robust and Simple approach. Iterate between:
1. Pick a random pair of 3 points on model & scan
2.Estimate alignment, and check for error.
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Guess and verify
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Can also refine the final result. Picks don’t have to be exact.

Guess and verify
Random Sampling (RANSAC)

A pair of triples (from $\mathbf{P}$ and $\mathbf{Q}$) are enough to determine a **rigid transform**, resulting in $O(n^3)$ RANSAC.

Surprisingly, a special set of 4 points, **congruent sets**, makes the problem simpler leading to $O(n^2)$!

Co-planar points remain coplanar

\[ r_1 = \frac{\|\mathbf{a-e}\|}{\|\mathbf{a-b}\|} \]

\[ r_2 = \frac{\|\mathbf{c-e}\|}{\|\mathbf{c-d}\|} \]

4-points Congruent Sets for Robust Surface Registration, Aiger et al., SIGGRAPH 2008
Method Overview

On the source shape, pick 4 (approx.) coplanar points.

Compute

\[ r_1 = \frac{\|a - e\|}{\|a - b\|} \quad r_2 = \frac{\|d - e\|}{\|d - c\|} \]

For every pair \((q_1, q_2)\) of points on the destination compute

\[ p_1 = q_1 + r_1(q_2 - q_1) \]
\[ p_2 = q_1 + r_2(q_2 - q_1) \]

Those pairs \((q_1, q_2), (q_3, q_4)\) for which \(p_1_{(q_1, q_2)} = p_2_{(q_3, q_4)}\) are a good candidate correspondence for \((a, b, c, d)\).

Under mild assumptions the procedure runs in \(O(n^2)\) time.
Method Overview

Can pick a few base points for partial matching.

Random sampling handles noise and outliers

4-points Congruent Sets for Robust Surface Registration, Aiger et al., SIGGRAPH 2008
Can pick a few base points for partial matching.

Partial matches

4-points Congruent Sets for Robust Surface Registration,
Aiger et al., SIGGRAPH 2008
Method Overview

Very robust to noise, fast global alignment.

Can further improve efficiency.

Still need to local refinement.

Can further reduce the number of candidates. **Not all points are created equal.**

Find salient, feature points.

4-points Congruent Sets for Robust Surface Registration, Aiger et al., SIGGRAPH 2008
## Global Matching – Approaches

Several classes of approaches:

1. Exhaustive Search
2. Normalization
3. Random Sampling
4. Invariance
Global Matching – Invariance

Try to characterize the shape using properties that are invariant under the desired set of transformations.

Conflicting interests – invariance vs. informativeness.

The most common pipeline:
1. identify salient feature points
2. compute informative and commensurable descriptors.
Matching Using Feature Points

1. Find **feature points** on the two scans (we’ll come back to that issue)
Approach

1. (Find feature points on the two scans)
2. Establish **correspondences**
Approach

1. (Find feature points on the two scans)
2. Establish correspondences
3. Compute the aligning transformation

Partially Overlapping Scans

Aligned Scans
Goal:
Identify when two points on different scans represent the same feature
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Are the surrounding regions similar?
Correspondence

Goal:
Identify when two points on different scans represent the same feature:
Are the surrounding regions similar?
Main Question

How to compare regions on the shape in an invariant manner?

A large variety of descriptors have been suggested.

To give an example, I’ll describe two.
Spin Images

Creates an image associated with a neighborhood of a point.

Compare points by comparing their spin images (2D).

Given a point and a normal, every other point is indexed by two parameters:

$\beta$ distance to tangent plane
$\alpha$ distance to normal line

Using Spin Images for Efficient Object Recognition in Cluttered 3D Scenes
Johnson et al, PAMI 99
Integral Volume Descriptor

Integral invariant signatures, Manay et al. ECCV 2004
Integral Invariants for Robust Geometry Processing, Pottmann et al. 2007-2009

\[ V_r(p) = \int_{B_r(p) \cap S} dx \]

Relation to mean curvature

\[ V_r(p) = \frac{2\pi}{3} r^3 - \frac{\pi H}{4} r^4 + O(r^5) \]

Robust Global Registration, Gelfand et al. 2005
Feature Based Methods

Once we have a feature descriptor, we can find the most unusual one: feature detection.

Establish correspondences by first finding reliable ones.

Propagate the matches everywhere.

To backtrack use branch-and-bound.

Robust Global Registration, Gelfand et al. 2005
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Deformable Shape Matching

No shape is completely deformable. Every deformable shape matching method uses some deformation model.

Local Deformable Matching:

- Each coordinate becomes an unknown. Use smoothness, elasticity constraints. Resulting problem is always non-linear.

Wand et al. SGP ’07

Li et al. SGP ’08

...
Deformable Shape Matching

Global/Local Deformable Matching Models:

Most common deformation model: **intrinsic isometries.** A deformation that preserves *geodesic* distances.

Recall that rigid deformations preserve *extrinsic* distances.

Isometric deformation model is useful in matching humans, animals, cloth, some molecular data, etc.
Related Work

Most global deformable matching methods use isometric deformation model.

GMDS

\[
\min_{\{y_1, \ldots, y_n\} \subseteq Y} \|d_X(x_i, x_j) - d_Y(y_i, y_j)\|
\]

Bronstein et al. PNAS ’06

Optimization is non-convex, performed on a small subset of points. Need an initial guess.
Related Work

Most global deformable matching methods use isometric deformation model.

GMDS

\[
\min_{\{y_1, \ldots, y_n\} \subseteq Y} \| d_X(x_i, x_j) - d_Y(y_i, y_j) \|
\]

Bronstein et al. PNAS ‘06

Find a few landmark correspondences, extend them using geodesic coordinates.

Huang et al. SGP ‘08
Deformable Shape Matching

To define descriptors identify what is preserved under the given class of deformations.

Isometric deformations preserve geodesic distances. And:

1. Gaussian Curvature at each point.
2. Laplace Beltrami Operator and its properties (eigenvalues, eigenvectors).
3. Heat Diffusion process.
Heat Equation on a Manifold

Heat kernel $k_t(x, y) : \mathbb{R}^+ \times \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$

$$f(x, t) = \int_{\mathcal{M}} k_t(x, y) f(y, 0) dy$$

$k_t(x, y)$: amount of heat transferred from $x$ to $y$ in time $t$. 

$t = 0.001$  $t = 0.02$  $t = 3$  $t = 40$
Heat Kernel Signature

\[ \text{HKS}(p) = k^M_t(p, p) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ : \text{amount of heat left at } p \text{ at time } t. \]

Signature of a point is a function of one variable.

Invariant to isometric deformations. Moreover complete:
Any continuous map between shapes that preserves HKS must preserve all distances.

A Concise and Provably Informative …,
Sun et al., SGP 2009
Conclusion

Shape Matching is an active area of research.

Local rigid matching works well. Many approaches to global matching. Work well depending on the domain.

Non-rigid matching is much harder. Isometric deformation model is common, and useful but limiting.

Research problems: other deformation models, consistent matching with many shapes, robust deformable matching.