Subdivision Surfaces

Bezier, Extraordinary Points, and Boundaries

Chaikin’s Corner Cutting Scheme

- Each point is cut by the ratio 1:3
- Converges to a quadratic B-Spline
Corner Cutting

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Corner Cutting

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Corner Cutting

Corner Cutting
Corner Cutting

Corner Cutting
Corner Cutting

Corner Cutting
Uniform B-Spline Curves - Reminder

- A set of control points $P_{1..n}$
- A degree $k$.
- A subdivision of the parameter domain $[0,1]$ into $n-m$ subdomains
  - Subdomain - a curve dependent on $k$ consecutive control points.
  - Knots – where the dependence shifts

Proof

- A uniform quadratic B-Spline on control points $P_{1..n}$ between any two consecutive knots can be parametrized:

$$P(t) = \begin{pmatrix} 1 & t & t^2 \end{pmatrix} \cdot \begin{pmatrix} P_k \\ P_{k+1} \\ P_{k+2} \end{pmatrix},$$

- For $0 \leq t \leq 1$

$$M = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$
B-Spline Refinement

• Suppose we look at a curve with only three points \( \{P_0, P_1, P_2\} \)
• The first half, reparametrized to be a full curve on its own:

\[
P\left(\frac{t}{2}\right) = \begin{pmatrix} 1 & \left(\frac{t}{2}\right) & \left(\frac{t}{2}\right)^2 \end{pmatrix} \cdot \begin{pmatrix} P_k \\ P_{k+1} \\ P_{k+2} \end{pmatrix}
\]

B-Spline Refinement

• Expressing the half curve as a full B-Spline on the parameter \( t \), with new control points \( Q \):

\[
P\left(\frac{t}{2}\right) = \begin{pmatrix} 1 & \left(\frac{t}{2}\right) & \left(\frac{t}{2}\right)^2 \end{pmatrix} \cdot \begin{pmatrix} P_k \\ P_{k+1} \\ P_{k+2} \end{pmatrix} = (1 + t^2) \cdot \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix} \cdot \begin{pmatrix} P_k \\ P_{k+1} \\ P_{k+2} \end{pmatrix}
\]

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Proof – cont’d

\[
(1 - t^2) \begin{pmatrix} 1 \n \frac{1}{2} \n \frac{1}{4} \end{pmatrix} M \begin{pmatrix} P_0 
 P_{a+1} 
 P_{a+2} \end{pmatrix} =
\]

\[
(1 - t^2) M^{-1} \begin{pmatrix} 1 \n \frac{1}{2} \n \frac{1}{4} \end{pmatrix} M \begin{pmatrix} P_0 
 P_{a+1} 
 P_{a+2} \end{pmatrix}
\]

Setting:

\[S = M^{-1} \begin{pmatrix} 1 \n \frac{1}{2} \n \frac{1}{4} \end{pmatrix} M = \frac{1}{4} \begin{pmatrix} 3 & 1 & 0 
 1 & 3 & 0 
 0 & 3 & 1 \end{pmatrix}\]

Proof – Cont’d

- We get:

\[
\frac{1}{4} \begin{pmatrix} 3 & 1 & 0 
 1 & 3 & 0 
 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} P_0 
 P_1 
 P_2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 3P_0 + P_1 
 P_0 + 3P_1 
 3P + P_2 \end{pmatrix} = \begin{pmatrix} Q_0 
 Q_1 
 Q_2 \end{pmatrix}
\]

- Similarly for the second half of the curve:

\[
\frac{1}{4} \begin{pmatrix} 3 & 1 & 0 
 1 & 3 & 0 
 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} P_0 
 P_1 
 P_2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 3P_1 + P_2 
 P_1 + 3P_2 
 P_1 + 3P_2 \end{pmatrix} = \begin{pmatrix} R_0 
 R_1 
 R_2 \end{pmatrix}
\]

- Notice that \( R_0 = Q_1 \) and \( R_1 = Q_2 \)
What we get

- 4 new independent control points
  - Exactly the result of Chaikin corner cutting!
- A uniform quadratic B-Spline curve similar to the first one, on these control points
  - The knot changes in the split parameter $t$.
- **Conclusion:** Chaikin’s corner cutting = uniform quadratic B-Spline refining!
  - This process is also called “Knot insertion”.

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Cubic Uniform B-Spline Refining

- Under the same reasoning:
Cubic Corner Cutting

Bezier curves by Subdivision

- Reminder: Bezier curves are produced by the DeCasteljau algorithm.
- Given parameter $t$, keep bisecting control polygon until converging to $C(t)$:

$$P_{AB} = (1-t)A + tB$$
Beziers Curves by Subdivision

- Curve can be subdivided into two curves at parameter value $t$
- Two new control polygons:
  - Most “left” and most “right” points

$$C_1 = \{P_{A}, P_{AB}, P_{ABC}, P_{ABCD}\}$$
$$C_2 = \{P_{ABCD}, P_{BCD}, P_{BCD}, P_{D}\}$$
- Proof is direct by DeCastleJau construction

Refined Control Polygon

- The two polygons join at $P_{ABCD}$
- The concatenation of both is a subdivided version of the original polygon!
- Keep subdividing both parts - joint control polygon converges to the smooth curve.
Catmull-Clark Subdivision

- Cubic Uniform corner cutting for tensor-product surfaces:
  
  - Regular vertices: valence 4
    - Grid of regular 3x3 cells reproduces cubic uniform B-Splines exactly

Extraordinary Vertices

- Valence != 4
- Catmull-Clark generalization:
Extraordinary Vertices

- A patch adjacent to an extraordinary vertex is subdivided into four patches
  - Three are regular, one stays irregular

Extraordinary Vertices

- Regions of “irregular effect” contract (Stam, 98’)
- Surface is $C^2$ Everywhere but in the limit points of this regions (where it is $C^1$)
Loop Subdivision

• Regular grids reproduce quartic “box splines”

• Regions of affect of extraordinary vertices contract as well

Boundary Vertices (and Creases)

• Catmull-Clark and Loop: boundaries chosen to reproduce cubic Spline curves

• Corner vertices (between two boundary pieces) are untouched