Signal and Image Processing by Computer (236327)

Winter 2015-16

Homework #1

Publication: November 12, 2015
Submission: by November 30 at 13:30

Guidelines:

- Submission is only in pairs.
- Submit your solution to the course box at Taub floor 1.
- Matlab code should be electronically submitted via the course website.
- You may answer in Hebrew or English, in a clear-handwrite or a printed form.
- Material that was taught in class can be used by referring to it without a proof. Other claims should be fully proved.
- Follow the FAQ at course website, as clarifications and corrections may be published only there.

להגשים:

תאריך ושעת הגשה: עד ה-30.11.2015 בשעה 13:30. הגשה בזוגות בלבד
הגשה ידנית לתא הקורס בטאוב קומה 1.
אף קוד המ ogóle יש להגיש אלכתוריית באירוט הקורס.
ניתן לענות בכתב ובפורמט מקורי, אך בائهم מאונכים.
אין זכות לגלץ דיבריו שלמרות אחרים או בתרגולים. משאירים לענות ולהגשה להימצאות הלימודים. יש להצביע על תשובות שלמה בלהביה ובקורסים.
.conv קובא את הנושא. המחברת וית-calendar יפורמים שפלי בל'גב

FAQ
Question #1 (25 points)
In class we considered the problem of scalar quantization, where a real valued variable \( x \in \mathbb{R} \) is quantized using \( b \) bits by the function
\[
Q_s(x) = r_i \quad \text{for} \quad x \in [d_{i-1}, d_i)
\]
where \( j = 2^b \), and \( \{r_i\}_{i=1}^J \) and \( \{d_i\}_{i=0}^J \) are the (scalar) representation values and decision levels, respectively.

We here extend the scalar problem to the case of vector quantization. Specifically, a real valued \( N \)-dimensional vector \( x \in \mathbb{R}^N \) is quantized using \( b \) bits by the function
\[
Q_v(x) = r_i \quad \text{for} \quad x \in D_i
\]
where \( \{r_i\}_{i=1}^J \) are \( N \)-dimensional representation vectors, and \( \{D_i\}_{i=1}^J \) are decision regions in \( \mathbb{R}^N \).

We consider here optimality in the MSE sense, defined as
\[
MSE = \int_{\mathbb{R}^N} \|x - Q_v(x)\|_2^2 \cdot p(x) dx
\]
where \( p(x) \) is the probability density function of the vector \( x \) (recall that \( p: \mathbb{R}^N \to [0,1] \)).

a. What are the optimal representation vectors \( \{r_i\}_{i=1}^J \), given the decision regions \( \{D_i\}_{i=1}^J \)?

b. What are the optimal decision regions \( \{D_i\}_{i=1}^J \), given the representation vectors \( \{r_i\}_{i=1}^J \)?

c. Write the Max-Lloyd algorithm for vector quantization.

d. Consider the vector quantization procedure (for \( x \in \mathbb{R}^N \)) that separately applies scalar Max-Lloyd quantization to each of the \( N \) components of the input.
What is the expected performance (i.e., in the sense of the resulting MSE as function of the utilized bit-budget) for an arbitrary input distribution of this vector quantizer compared to the one of section c? Explain.
**Question #2 (25 points)**

Consider the following signal for $t \in [0,1]$:

$$\varphi(t) = A \cdot \sin(2\pi \omega t)$$

where $A$ and $\omega$ are the signal’s amplitude and frequency parameters, respectively. We assume here that $\omega$ is an integer.

We would like to find the optimal bit-allocation for $\varphi(t)$.

a. Express the derivative-energy ($\text{Energy}(\varphi')$) and the value-range ($\varphi_H - \varphi_L$) of $\varphi(t)$, as required for the bit-allocation optimization.

b. Find the optimal number of samples ($N_t$) and quantization bits ($b$) under the constraint of overall bit-budget $B$.
   
   i. Formulate the bit-allocation optimization problem.
   
   ii. Develop the problem to optimization on a single variable (i.e., $b$ or $N_t$), and formulate the mathematical expression (equation) that defines optimality.

c. You are given a total bit-budget of $B = 200$ bits. Compare (and explain) the values of the optimal bit-allocation parameters (i.e., $N_t$ and $b$) of the following signals:
   
   i. $\varphi_1(t) = 5 \cdot \sin(2\pi t)$.
   
   ii. $\varphi_2(t) = 5 \cdot \sin(20\pi t)$.

   Here you can use Matlab for numerically solving the optimality equation from section (b.ii). Useful Matlab functions here are ‘solve’ and ‘lambertw’. Note that ‘solve’ may return an expression that mathematically depends on the ‘lambertw’ function, so, in turn, you can get a real valued result by appropriately applying the ‘lambertw’ Matlab function.
**Question #3 (20 points)**

Let us define a 3D (grayscale) video signal by a continuous function \( \varphi(x, y, t) \), where

\[
\varphi : [0,1] \times [0,1] \times [0,1] \to \mathbb{R}.
\]

1. a. What are the optimal sampling coefficients, before quantization, for this signal (in the mean squared error sense)?
   b. What is the MSE value for each 3D sampling area for the optimal coefficients?

2. Assume a uniformly distributed input and a uniform quantizer. What is the quantization mean squared error for \( b \) bits?

3. A video is a 3D signal that can be represented as a sequence of 2D images, each representing a single frame. Similar to what we have seen in class for 1D and 2D signals, we can perform sampling and quantization on the signal.
   Write the expression for the bit-allocation MSE in this 3D case.
   In this subsection you may skip the thorough proof and satisfy with good explanation and reasoning.

4. You get \( B \) bits for the digitization procedure.
   Write the expression for the bit-budget constraint for the 3D case.
   Explain the meaning of each one of the parameters you use.

5. You are given two video signals of 88 frames of 144x176 size each.
   Eight consecutive frames are presented in the next page for each video, you can assume that these videos do not change significantly in terms of their properties in the other frames.
Video 1: A news presenter.

Video 2: A tennis player in a game.

An optimal bit-allocation has been performed on the above videos, given a budget of $B = 2230272$ bits.

The optimal bit-allocation result for one of the above videos was $N_x = 88, N_y = 72, N_z = 88, b = 4$, while the optimal bit allocation result for the other video was $N_x = 176, N_y = 72, N_z = 22, b = 8$. Match between the videos and the results. Explain your choice.
Matlab Part (30 points)

Instructions:

- In this part use a grayscale image (of 256 graylevels) and of a size of at least 256x256 pixels.
  You can transform RGB-color image to a grayscale image using the matlab function rgb2gray.
- Along this part show the image in the same size and dynamic range (pass [0,255] to the imshow function).
- Figures should be appropriately titled.
- Your written report should be submitted with the dry part, whereas the code should be electronically submitted.

1. We would like to estimate the probability-density-function of the graylevels in the image using the image histogram. This can be calculated using the Matlab function imhist (Note that imhist is sensitive to the graylevel range, e.g. [0,1], [0,255], etc.). If the histogram seems too uniform, please pick another image with a non-uniform distribution.

2. Apply uniform quantization on the image using $b$ bits per pixel.
   a. Show the MSE as a function of the bit-budget $b$ for $b = 1, \ldots, 8$.
   b. Plot the decision and representation levels for representative $b$ values.

3. Implement the Max-Lloyd quantizer.
   a. Show the MSE as a function of the bit-budget $b$ (for $b = 1, \ldots, 8$).
   b. Plot the decision and representation levels for representative $b$ values.
   c. Compare the results to those of the uniform quantizer. Explain the differences.