Signal and Image Processing (236327)

Tutorial 2

Sampling
Signal Digitization: Motivation

• “Real-world” signals
  • Defined on a continuous area
  • Have values from a continuous and unbounded range

• Digital representation
  • Discrete and finite area and values
    • I.e., finite-length vectors of discrete values from a finite range.
  • Required for
    • Processing
    • Storage
    • Transmission
    • Presentation

that involve computers and other digital devices/media.
Signal Sampling

Transfers a signal from a continuous domain to a discrete one:

Input:
A continuous function \( \varphi(t) \) defined on the continuous interval \([0,1]\)

Output:
• A function \( \varphi[n] \) defined on the discrete domain \( n = 1, \ldots, N \)
• Also formed as a finite length vector of \( N \) elements:

\[
\begin{bmatrix}
\varphi_1 \\
\varphi_2 \\
\varphi_3 \\
\vdots \\
\varphi_N
\end{bmatrix}
\]
Reconstruction from Sampling

Forming a continuous signal from the discrete (sampled) representation:

Input:
• A function $\varphi[n]$ defined on the discrete domain $n = 1, \ldots, N$

• Also formed as a finite length vector of $N$ elements:
  \[ \varphi_1 \varphi_2 \varphi_3 \ldots \varphi_N \]

Output:
A continuous function $\hat{\varphi}(t)$ defined on the continuous interval $[0,1]$

The above is a simple reconstruction method.
The Sampling Error

The sampling-reconstruction process introduces an error:

\[ e(t) = \varphi(t) - \hat{\varphi}(t) \]

The error is the difference between the reconstruction and the continuous input-signal (defined for any \( t \in [0,1] \)):

The error-level is affected by:

- The sampling procedure
- The reconstruction method
- The signal properties
Optimal Sampling

• Consider **uniform sampling** of \( N \) samples in the area \([0,1]\):
  • All the sampling intervals have an **equal (fixed) size of** \( \Delta = \frac{1}{N} \).
  • The sampling intervals are \( \Delta_i = \left[ \frac{i-1}{N}, \frac{i}{N} \right] \text{ for } i = 1, \ldots, N \)

• Each sampling-interval is represented using a single value \( \varphi_i \).

• **Optimal sampling-coefficients**
  • Optimality is in terms of **objective error criterion** \( C\{e(t)\} \)
  • \( C\{e(t)\} \) evaluates a given error signal using a non-negative scalar value, that should be minimized:
    \[
    \min_{\varphi_1, \ldots, \varphi_N} C\{e(t)\}
    \]

  • Examples for \( C\{\cdot\} \): **Mean-Squared-Error (MSE)**, **Mean-Absolute-Deviation (MAD)**
Optimal Sampling in the Mean-Squared-Error (MSE) Sense

\[ MSE_{total} = \frac{1}{N} \sum_{i=1}^{N} MSE_i \]

where the MSE of the \( i^{th} \) interval is \[ MSE_i = \frac{1}{\Delta} \int_{(i-1)\Delta}^{i\Delta} [\varphi(t) - \varphi_i]^2 dt \]

Finding the optimal \( \varphi_i \) by requiring \( \frac{d}{d\varphi_i} MSE_i = 0 \):

\[ \frac{d}{d\varphi_i} MSE_i = -\frac{2}{\Delta} \int_{(i-1)\Delta}^{i\Delta} [\varphi(t) - \varphi_i] \, dt = -\frac{2}{\Delta} \left[ \int_{(i-1)\Delta}^{i\Delta} \varphi(t) \, dt - \int_{(i-1)\Delta}^{i\Delta} \varphi_i \, dt \right] = -\frac{2}{\Delta} \left[ \int_{(i-1)\Delta}^{i\Delta} \varphi(t) \, dt - \Delta \cdot \varphi_i \right] \]

\[ \rightarrow \varphi_i = \frac{1}{\Delta} \int_{(i-1)\Delta}^{i\Delta} \varphi(t) \, dt \]

The MSE-optimal coefficients are the interval averages.
Optimal Sampling in the Mean-Squared-Error (MSE) Sense

Exercise #1:

The signal \( \varphi(t) = \begin{cases} t, & \text{for } t \in [0,1] \\ 0, & \text{otherwise} \end{cases} \)

is uniformly sampled to have \( N \) samples.

What are the optimal coefficients (in the MSE sense) ?

The \( i^{th} \) interval is \( \Delta_i = \left[ \frac{i-1}{N}, \frac{i}{N} \right] \)

\[
\varphi_i = \frac{1}{\Delta} \int_{(i-1)\Delta}^{i\Delta} t \, dt = \frac{1}{\Delta} \cdot \frac{t^2}{2} \bigg|_{(i-1)\Delta}^{i\Delta} = \frac{\Delta^2}{\Delta} \cdot \frac{i^2 - (i-1)^2}{2} = \Delta \cdot \left( i - \frac{1}{2} \right)
\]
Optimal Sampling in the Mean-Squared-Error (MSE) Sense

Exercise #1:

The signal \( \varphi(t) = \begin{cases} t, & \text{for } t \in [0,1] \\ 0, & \text{otherwise} \end{cases} \)
is uniformly sampled to have \( N \) samples.

What are the MSE in the intervals?

\[
MSE_i = \frac{1}{\Delta} \int_{(i-1)\Delta}^{i\Delta} [t - \varphi_i]^2 dt = \frac{1}{\Delta} \int_{(i-1)\Delta}^{i\Delta} \left[ t - \Delta \cdot \left( i - \frac{1}{2} \right) \right]^2 dt
\]

Change of integration variables, \( \tilde{t} = t - \left( i - \frac{1}{2} \right) \Delta \), leads to

\[
= \frac{1}{\Delta} \int_{\frac{\Delta}{2}}^{\frac{3\Delta}{2}} \tilde{t}^2 d\tilde{t} = \frac{1}{\Delta} \left[ \frac{\tilde{t}^3}{3} \right]_{\frac{\Delta}{2}}^{\frac{3\Delta}{2}} = \frac{\Delta^2}{12}
\]
Optimal Sampling in the Mean-Absolute-Deviation (MAD) Sense

\[ \text{MAD}_{\text{total}} = \frac{1}{N} \sum_{i=1}^{N} \text{MAD}_i \]

where the MAD of the \(i^{th}\) interval is \( \text{MAD}_i = \frac{1}{\Delta} \int_{(i-1)\Delta}^{i\Delta} |\varphi(t) - \varphi_i| dt \)

Finding the optimal \(\varphi_i\):

\[ \frac{d}{d\varphi_i} \text{MAD}_i = \frac{1}{\Delta} \int_{(i-1)\Delta}^{i\Delta} \frac{d}{d\varphi_i} |\varphi(t) - \varphi_i| dt = -\frac{1}{\Delta} \int_{t=(i-1)\Delta}^{i\Delta} \text{sign}(\varphi(t) - \varphi_i) dt \]

where we used \( \frac{d}{dx} |x| = \text{sign}(x) \), \( \text{sign}(x) \triangleq \begin{cases} 1 & , x > 0 \\ 0 & , x = 0 \\ -1 & , x < 0 \end{cases} \)
Optimal Sampling in the Mean-Absolute-Deviation (MAD) Sense

Finding the optimal $\varphi_i$:

$$\frac{d}{d\varphi_i} \text{MAD}_i = \frac{1}{\Delta} \int_{(i-1)\Delta}^{i\Delta} \frac{d}{d\varphi_i} \left| \varphi(t) - \varphi_i \right| dt = \frac{1}{\Delta} \int_{t=(i-1)\Delta}^{i\Delta} \text{sign}(\varphi(t) - \varphi_i) dt$$

$$= \frac{1}{\Delta} \int_{\varphi(t)>\varphi_i} 1 \ dt + \frac{1}{\Delta} \int_{\varphi(t)<\varphi_i} (-1) dt + \frac{1}{\Delta} \int_{\varphi(t)=\varphi_i} 0 \ dt$$

Asserting zero-derivative gives, the optimality criterion for $\varphi_i$:

$$\int_{\varphi(t)>\varphi_i} dt = \int_{\varphi(t)<\varphi_i} dt$$

*The MAD-optimal coefficients are the interval medians.*
Optimal Sampling in the MAD Sense

Exercise #2:

The signal \( \varphi(t) = \begin{cases} t, & \text{for } t \in [0,1] \\ 0, & \text{otherwise} \end{cases} \)
is uniformly sampled to have \( N \) samples.

What are the optimal coefficients (in the MAD sense) ?

The \( i^{th} \) interval is \( \Delta_i = [\Delta(i-1), \Delta i] \)

\[ \varphi_i = \text{median}\{\varphi(t), t \in [\Delta(i-1), \Delta i]\} = \text{median}\{t \in [\Delta(i-1), \Delta i]\} = \Delta \cdot \left( i - \frac{1}{2} \right) \]

\[ \text{MAD}_i = \frac{1}{\Delta} \int_{(i-1)\Delta}^{i\Delta} \left| t - \Delta \cdot \left( i - \frac{1}{2} \right) \right| dt = \frac{1}{\Delta} \int_{\Delta/2}^{\Delta/2} |\tilde{t}| d\tilde{t} = \frac{1}{\Delta} \cdot 2 \int_{0}^{\Delta/2} \tilde{t} d\tilde{t} = \frac{1}{\Delta} \cdot 2 \cdot \frac{\Delta^2}{8} = \frac{\Delta}{4} \]
Optimal Sampling: MSE vs. MAD

Exercise #3:

The signal \( \varphi(t) = \begin{cases} 1, & t \in [0, \frac{1}{4}] \\ \frac{4}{3}(1 - t), & t \in \left[ \frac{1}{4}, 1 \right] \\ 0, & \text{otherwise} \end{cases} \)

is uniformly sampled to have \( N = 2 \) samples (\( \Delta = \frac{1}{2} \)).

What are the optimal coefficients (in the **MSE** sense) ?

\[
\varphi_2^{MSE} = \frac{1}{\Delta} \int_{1/2}^{1} \frac{4}{3}(1 - t)dt = 2 \cdot \frac{4}{3}t - \frac{4t^2}{3} \bigg|_\frac{1}{2} = \frac{1}{3}
\]

\[
\varphi_1^{MSE} = \frac{1}{\Delta} \int_{0}^{1/2} \varphi(t)dt = \frac{1}{\Delta} \int_{0}^{1/4} 1dt + \frac{1}{\Delta} \int_{1/4}^{1/2} \frac{4}{3}(1 - t)dt = \frac{1}{2} + \frac{5}{12} = \frac{11}{12}
\]
Optimal Sampling: MSE vs. MAD

The signal $\varphi(t) = \begin{cases} 1, & t \in \left[0, \frac{1}{4}\right] \\ \frac{4}{3}(1-t), & t \in \left[\frac{1}{4}, 1\right] \\ 0, & \text{otherwise} \end{cases}$

is uniformly sampled to have $N = 2$ samples ($\Delta = \frac{1}{2}$).

What are the optimal coefficients (in the MAD sense)?

$$\varphi_{2}^{MAD} = \text{median}\left\{ \varphi(t), t \in \left[\frac{1}{2}, 1\right] \right\} = \text{median}\left\{ \frac{4}{3}(1-t), t \in \left[\frac{1}{2}, 1\right] \right\} = \frac{1}{3}$$

$$\varphi_{1}^{MAD} = \text{median}\left\{ \varphi(t), t \in \left[0, \frac{1}{2}\right] \right\} = \text{median}\left\{ \left\{ 1, t \in \left[0, \frac{1}{4}\right] \right\} \cup \left\{ \frac{4}{3}(1-t), t \in \left[\frac{1}{4}, \frac{1}{2}\right] \right\} \right\} = 1$$
Resampling

• Consider a discrete (already sampled) signal
• This signal can be sub-sampled to contain less samples
  • i.e., to be represented by a shorter vector.

\[ M = \frac{N}{D} \]

This case is similar to the sampling of a continuous signal, however it has a discrete nature:

\( D \) is the sub-sampling factor, assumed to yield an integer \( M \).
Resampling

\[ \text{MSE}_{total} = \frac{1}{M} \sum_{i=1}^{M} \text{MSE}_i \]

where the MSE of the \( i^{th} \) interval \( (i = 1, \ldots, M) \) is

\[ \text{MSE}_i = \frac{1}{D} \sum_{j=(i-1)D+1}^{iD} (\varphi_j - \bar{\varphi}_i)^2 \]

Similar to the continuous case, the optimal coefficient is the average of the “interval”:

\[ \bar{\varphi}_i = \frac{1}{D} \sum_{j=(i-1)D+1}^{iD} \varphi_j \]

A simple reconstruction procedure may be applied by

\[ \hat{\varphi}_j = \bar{\varphi}_k \quad \text{where } k = \left\lfloor \frac{j}{D} \right\rfloor + 1 \quad \text{(for } j = 1, \ldots, N) \]