Signal and Image Processing by Computer (236327)

Spring 2017

Homework #3 - Solution

Question #2 (10 points)

Let us consider a real-valued unitary matrix $U$ of size $N \times N$, formed from $N$ column-vectors $\{u_i\}_{i=1}^{N}$. The direct-transform of some signal-vector $x$ is considered here as $U^T x$ (note the transpose).

An orthonormal set of $N$ continuous functions can be constructed as:

$$\psi_i(t) = \sum_{k=1}^{N} u_i^{(k)} \cdot \psi_k^S(t), i = 1, ..., N$$

where,

$u_i^{(k)}$ is the $k^{th}$ element of the $i^{th}$ column-vector $u_i$

$\psi_k^S(t)$ is the $k^{th}$ function of the standard family (for $N$ samples), defined as

$$\psi_k^S(t) = \begin{cases} \sqrt{N} & \text{for } t \in \left[\frac{i-1}{N}, \frac{i}{N}\right] \\ 0 & \text{otherwise} \end{cases}$$

Prove that the orthonormality of the columns, $\{u_i\}_{i=1}^{N}$, leads to orthonormality of the functions $\{\psi_i(t)\}_{i=1}^{N}$.

Solution:

Let us show first that the (real-valued) functions $\{\psi_i(t)\}_{i=1}^{N}$ are normalized:
\[ \langle \psi_i(t), \psi_i(t) \rangle = \int_{t=0}^{1} \psi_i^2(t) dt = \int_{t=0}^{1} \left( \sum_{k=1}^{N} u_i^{(k)} \cdot \psi_k^x(t) \right)^2 dt \]

\[ = \int_{t=0}^{1} \sum_{k=1}^{N} \sum_{l=1}^{N} u_i^{(k)} \cdot \psi_k^x(t) \cdot u_i^{(l)} \cdot \psi_l^x(t) dt \]

\[ = \sum_{k=1}^{N} \sum_{l=1}^{N} u_i^{(k)} u_l^{(l)} \int_{t=0}^{1} \psi_k^x(t) \psi_l^x(t) dt = \sum_{k=1}^{N} \sum_{l=1}^{N} u_i^{(k)} u_l^{(l)} \delta_{k,l} = \sum_{k=1}^{N} u_i^{(k)} u_i^{(k)} \]

\[ = u_i^T u_i = 1 \]

For the case where \( i \neq j \):

\[ \langle \psi_i(t), \psi_j(t) \rangle = \int_{t=0}^{1} \psi_i(t) \psi_j(t) dt = \int_{t=0}^{1} \left( \sum_{k=1}^{N} u_i^{(k)} \cdot \psi_k^x(t) \right) \left( \sum_{l=1}^{N} u_j^{(l)} \cdot \psi_l^x(t) \right) dt \]

\[ = \int_{t=0}^{1} \sum_{k=1}^{N} \sum_{l=1}^{N} u_i^{(k)} \cdot \psi_k^x(t) \cdot u_j^{(l)} \cdot \psi_l^x(t) dt \]

\[ = \sum_{k=1}^{N} \sum_{l=1}^{N} u_i^{(k)} u_j^{(l)} \int_{t=0}^{1} \psi_k^x(t) \psi_l^x(t) dt = \sum_{k=1}^{N} \sum_{l=1}^{N} u_i^{(k)} u_j^{(l)} \delta_{k,l} = \sum_{k=1}^{N} u_i^{(k)} u_j^{(k)} \]

\[ = u_i^T u_j = 0 \]

In the above development we used the orthonormality of the functions \( \{ \psi_i^x(t) \}_{i=1}^{N} \) and the orthonormality of the \( \{ u_i \}_{i=1}^{N} \), and also that summation and integration are can replace their order (due to their linearity).

**Question #3 (15 points) – Bandlimited Signals**

A given signal \( \varphi(t) \), defined over the interval \([0, 1]\), is \( N_0 \)-bandlimited, implying that its projections onto the Fourier family functions hold

\[ \langle \varphi(t), e^{i2\pi k t} \rangle = 0 \quad \text{for} \quad |k| > N_0 \]

Recall that \( k \) is an integer that can be positive/negative/zero, and that the inner product is defined here as
\[ \langle \varphi(t), e^{i2\pi kt} \rangle = \int_0^1 \varphi(t)e^{-i2\pi kt} dt \]

The signal \( \varphi(t) \) is going through the following smoothing operation resulting in the signal \( \varphi_{|\Delta|}^{\text{smooth}}(t) \), defined for \( t \in [0,1) \), via the relation

\[ \varphi_{|\Delta|}^{\text{smooth}}(t) = \frac{1}{|\Delta|} \int_t^{t+|\Delta|} \varphi_p(\xi) d\xi \]

where \( |\Delta| \in (0,1) \) is the smoothing interval size, and \( \varphi_p(t) \) is the periodic extension of \( \varphi(t) \) defined as

\[ \varphi_p(t + m) = \varphi(t) \quad \text{for} \quad t \in [0,1) \quad \text{and any integer} \quad m \]

Prove that the signal \( \varphi_{|\Delta|}^{\text{smooth}}(t) \) is also \( N_0 \)-bandlimited.

**Solution:**

Please note that the maximal points reduced for this question is 5 and not 15 (meaning that 10 points were given anyway).

We present here two approaches for solving this question.

**Solution Approach #1:**

\[ \varphi_{|\Delta|}^{\text{smooth}}(t) = \frac{1}{|\Delta|} \int_t^{t+|\Delta|} \varphi_p(\xi) d\xi = \int_{\xi=-\infty}^{\infty} \varphi_p(\xi) h(t - \xi) d\xi = \int_{\xi=-\infty}^{\infty} \varphi_p(t - \xi) h(\xi) d\xi \]

where

\[ h(t) = \begin{cases} 
\frac{1}{|\Delta|}, & \text{for} \ t \in [-|\Delta|, 0) \\
0, & \text{else} 
\end{cases} \]

\[ \langle \varphi_{|\Delta|}^{\text{smooth}}(t), e^{i2\pi kt} \rangle = \int_0^1 \varphi_{|\Delta|}^{\text{smooth}}(t)e^{-i2\pi kt} dt = \int_{t=0}^{1} \int_{\xi=-\infty}^{\infty} \varphi_p(t - \xi) h(\xi) d\xi \cdot e^{-i2\pi kt} d\xi \\
= \int_{\xi=-\infty}^{\infty} \left( \int_{t=0}^{1} \varphi_p(t - \xi) e^{-i2\pi kt} dt \right) h(\xi) d\xi = (*) \]

Integration variable substitution to \( \tilde{\xi} = t - \xi \). Then
\[ (*) = \int_{\xi = -\infty}^{\infty} \left( \int_{\hat{t} = -\xi}^{1-\xi} \varphi_p(\hat{t}) e^{-i2\pi k(\hat{t} + \xi)} d\hat{t} \right) h(\xi) d\xi = \int_{\xi = -\infty}^{\infty} \left( \int_{\hat{t} = -\xi}^{1-\xi} \varphi_p(\hat{t}) e^{-i2\pi nk \hat{t}} d\hat{t} \right) e^{-i2\pi nk \xi} h(\xi) d\xi = (***) \]

Noting that
\[ \int_{\hat{t} = -\xi}^{1-\xi} \varphi_p(\hat{t}) e^{-i2\pi nk \hat{t}} d\hat{t} = \int_{\hat{t} = -\xi}^{0} \varphi_p(\hat{t}) e^{-i2\pi nk \hat{t}} d\hat{t} + \int_{\hat{t} = 0}^{1-\xi} \varphi_p(\hat{t}) e^{-i2\pi nk \hat{t}} d\hat{t} = (.) \]

Due to periodicity (with period length 1): \( \varphi_p(\hat{t} + 1) = \varphi_p(\hat{t}) \) and \( e^{-i2\pi k(\hat{t} + 1)} = e^{-i2\pi nk \hat{t}} \) then
\[ (.) = \int_{\hat{t} = -\xi}^{0} \varphi_p(\hat{t} + 1) e^{-i2\pi nk(\hat{t} + 1)} d\hat{t} + \int_{\hat{t} = 0}^{1-\xi} \varphi_p(\hat{t}) e^{-i2\pi nk \hat{t}} d\hat{t} = \int_{\hat{t} = 1-\xi}^{1} \varphi_p(\hat{t}) e^{-i2\pi nk \hat{t}} d\hat{t} + \int_{\hat{t} = 0}^{1-\xi} \varphi_p(\hat{t}) e^{-i2\pi nk \hat{t}} d\hat{t} = \int_{t = 0}^{1} \varphi(t) e^{-i2\pi nk t} dt = \langle \varphi(t), e^{i2\pi nk t} \rangle \]

Now, setting the last result in (**) gives
\[ (***) = \int_{\xi = -\infty}^{\infty} \langle \varphi(t), e^{i2\pi nk t} \rangle \cdot e^{-i2\pi nk \xi} h(\xi) d\xi \]

that for \( |k| > N_0 \) leads to
\[ \int_{\xi = -\infty}^{\infty} 0 \cdot e^{-i2\pi nk \xi} h(\xi) d\xi = 0 \]

and therefore, \( \langle \varphi_{\text{smooth}}(t), e^{i2\pi nk t} \rangle = 0 \) for \( |k| > N_0 \) meaning that \( \varphi_{\text{smooth}}(t) \) is \( N_0 \)-bandlimited.
Solution Approach #2:

Since $\varphi(t)$ is $N_0$-bandlimited:

$$\langle \varphi(t), e^{i2\pi kt} \rangle = 0 \text{ for } |k| > N_0$$

and therefore we can represent the signal $\varphi(t)$, for $t \in [0,1)$, as

$$\varphi(t) = \sum_{k=-N_0}^{N_0} \langle \varphi(t), e^{i2\pi kt} \rangle e^{i2\pi kt}$$

The periodic extension of the signal was defined as

$$\varphi_p(t + m) = \varphi(t) \text{ for } t \in [0,1) \text{ and any integer } m$$

and one can notice that the periodic extension also holds (for $t \in [0,1)$ and any integer $m$)

$$\varphi_p(t + m) = \varphi(t) = \sum_{k=-N_0}^{N_0} \langle \varphi(t), e^{i2\pi kt} \rangle e^{i2\pi kt} = \sum_{k=-N_0}^{N_0} \langle \varphi(t), e^{i2\pi kt} \rangle e^{i2\pi km}$$

where we relied on the fact that $e^{i2\pi km} = 1$ for an integer $m$ (and integer $k$).

Therefore, we can write for any $t$:

$$\varphi_p(t) = \sum_{k=-N_0}^{N_0} \langle \varphi(t), e^{i2\pi kt} \rangle e^{i2\pi kt}$$

Let us consider $\varphi_{\text{smooth}}^{|\Delta|}(t)$ as

$$\varphi_{\text{smooth}}^{|\Delta|}(t) = \frac{1}{|\Delta|} \int_{t}^{t+|\Delta|} \varphi_p(\xi) d\xi = \frac{1}{|\Delta|} \sum_{k=-N_0}^{N_0} \langle \varphi(t), e^{i2\pi k\xi} \rangle e^{i2\pi k\xi} d\xi$$

$$= \frac{1}{|\Delta|} \sum_{k=-N_0}^{N_0} \left( \langle \varphi(t), e^{i2\pi k\xi} \rangle \frac{1}{i2\pi k} e^{i2\pi k\xi} \right)$$

$$= \frac{1}{|\Delta|} \sum_{k=-N_0}^{N_0} \left( \langle \varphi(t), e^{i2\pi k\xi} \rangle \frac{1}{i2\pi k} \right) e^{i2\pi k\xi}$$
$$\frac{1}{|\Delta|} \sum_{k=-N_0}^{N_0} \langle \varphi(t), e^{i2\pi kt} \rangle \cdot \frac{1}{i2\pi k} (e^{i2\pi k(t+|\Delta|)} - e^{i2\pi kt})$$

$$= \sum_{k=-N_0}^{N_0} \langle \varphi(t), e^{i2\pi kt} \rangle \cdot \frac{1}{i2\pi k} (e^{i2\pi k|\Delta|} - 1) e^{i2\pi kt}$$

$$= \sum_{k=-N_0}^{N_0} \left( \langle \varphi(t), e^{i2\pi kt} \rangle \cdot \frac{i(1 - e^{i2\pi k|\Delta|})}{2\pi k|\Delta|} \right) e^{i2\pi kt}$$

The last expression for $\varphi_{\text{smooth}}(t)$ shows its (exact) representation using Fourier functions with $|k| \leq N_0$, implying $\varphi_{\text{smooth}}(t)$ is $N_0$-bandlimited. This can be more clearly observed if we define the coefficients

$$\gamma_k = \langle \varphi(t), e^{i2\pi kt} \rangle \cdot \frac{i(1 - e^{i2\pi k|\Delta|})}{2\pi k|\Delta|} , \quad k = -N_0, \ldots, N_0$$

then these $\gamma_k$ (scalar) values are the Fourier coefficients in the representation:

$$\varphi_{\text{smooth}}(t) = \sum_{k=-N_0}^{N_0} \gamma_k e^{i2\pi kt}$$

**Question #4 (15 points) – Linearity and Shift-Invariance of Operators**

Consider a signal $\varphi(t)$ that is defined over the interval $[0,1)$.

The periodic extension of $\varphi(t)$ is

$$\varphi_p(t + m) = \varphi(t) \quad \text{for} \ t \in [0,1) \ \text{and any integer} \ m$$

a. The generalized smoothing operation, with respect to an arbitrary smoothing-weight real-valued function $w(\cdot, \cdot)$ is defined, for $t \in [0,1)$, as

$$\varphi^\text{opt}(t) = \int_{-\infty}^{\infty} w(\xi, t) \varphi_p(\xi) \, d\xi$$

o Is the above operation linear? Prove.

o Is the above operation shift-invariant? Prove.

**Solution:**

**Linearity:**

Let us show that the above operator is linear:
By the definition of the operator:

\[ \mathcal{H}\{\varphi_p(t)\} = \int_{-\infty}^{\infty} w(\xi, t) \varphi_p(\xi) \, d\xi \]

Then, for an input formed as a linear combination of two signals \( \varphi_p^{(1)}(t) \) and \( \varphi_p^{(2)}(t) \) we get

\[ \mathcal{H}\{a \cdot \varphi_p^{(1)}(t) + b \cdot \varphi_p^{(2)}(t)\} = \int_{-\infty}^{\infty} w(\xi, t) \left[a \cdot \varphi_p^{(1)}(\xi) + b \cdot \varphi_p^{(2)}(\xi)\right] \, d\xi \]

\[ = a \int_{-\infty}^{\infty} w(\xi, t) \varphi_p^{(1)}(\xi) \, d\xi + b \int_{-\infty}^{\infty} w(\xi, t) \varphi_p^{(2)}(\xi) \, d\xi \]

\[ = a \cdot \mathcal{H}\{\varphi_p^{(1)}(t)\} + b \cdot \mathcal{H}\{\varphi_p^{(2)}(t)\} \]

Hence, the system is linear, as expected from the linearity of the integral.

**Shift invariance:**

The shifting operator \( T_{t_0} \) is applied on the input signal, forming the shifted signal \( \varphi_p^{\text{shifted}}(t) = T_{t_0}\{\varphi_p(t)\} = \varphi_p(t - t_0) \). Then, the system output for a shifted input is

\[ \mathcal{H}\left\{T_{t_0}\{\varphi_p(t)\}\right\} = \mathcal{H}\{\varphi_p^{\text{shifted}}(t)\} = \int_{-\infty}^{\infty} w(\xi, t) \varphi_p^{\text{shifted}}(\xi) \, d\xi \]

\[ = \int_{-\infty}^{\infty} w(\xi, t) \varphi_p(\xi - t_0) \, d\xi \]

The shifted system output (for a non-shifted input) is

\[ T_{t_0}\left\{\mathcal{H}\{\varphi_p(t)\}\right\} = T_{t_0}\left\{\int_{-\infty}^{\infty} w(\xi, t) \varphi_p(\xi) \, d\xi\right\} = \int_{-\infty}^{\infty} w(\xi, t - t_0) \varphi_p(\xi) \, d\xi \]

In order to compare \( \mathcal{H}\left\{T_{t_0}\{\varphi_p(t)\}\right\} \) and \( T_{t_0}\left\{\mathcal{H}\{\varphi_p(t)\}\right\} \) we will bring \( \mathcal{H}\left\{T_{t_0}\{\varphi_p(t)\}\right\} \) to the form of integration with respect to a non-shifted \( \varphi_p(\cdot) \):

\[ \mathcal{H}\left\{T_{t_0}\{\varphi_p(t)\}\right\} = \int_{-\infty}^{\infty} w(\xi, t) \varphi_p(\xi - t_0) \, d\xi = \int_{-\infty}^{\infty} w(\xi + t_0, t) \varphi_p(\xi) \, d\xi \]

where we applied the integration-variable substitution \( \xi = \xi - t_0 \).

Now we can observe that

\[ \mathcal{H}\left\{T_{t_0}\{\varphi_p(t)\}\right\} = \int_{-\infty}^{\infty} w(\xi + t_0, t) \varphi_p(\xi) \, d\xi \neq \int_{-\infty}^{\infty} w(\xi, t - t_0) \varphi_p(\xi) \, d\xi = T_{t_0}\left\{\mathcal{H}\{\varphi_p(t)\}\right\} \]

and therefore the system is **not shift-invariant**.
b. Consider the operator applied on the signal \( \varphi(t) \) as its convolution with the real-valued function \( h(\cdot) \), namely,
\[
\varphi^{op}(t) = \int_{-\infty}^{\infty} h(t-\xi) \varphi_p(\xi) \, d\xi
\]

- Is the above operation linear? Prove.
- Is the above operation shift-invariant? Prove.

**Solution:**
The system here is a particular case of the system from subsection (a). We can see it by defining the kernel function \( w \) from to be
\[
w(\xi, t) = h(t-\xi)
\]
Therefore, we immediately get that the system here is also linear, also proved as
\[
\mathcal{H} \left\{ a \cdot \varphi_p^{(1)}(t) + b \cdot \varphi_p^{(2)}(t) \right\} = a \int_{-\infty}^{\infty} h(t-\xi) \varphi_p^{(1)}(t) \, d\xi + b \int_{-\infty}^{\infty} h(t-\xi) \varphi_p^{(2)}(t) \, d\xi
\]
\[
= a \cdot \mathcal{H} \{ \varphi_p^{(1)}(t) \} + b \cdot \mathcal{H} \{ \varphi_p^{(2)}(t) \}
\]

As for the shift-invariance property we return to the expression obtained in subsection (a) and set there \( w(\xi, t) = h(t-\xi) \):
\[
\mathcal{H} \{ T_{t_0} \{ \varphi_p(t) \} \} = \int_{-\infty}^{\infty} w(\xi + t_0, t) \varphi_p(\xi) \, d\xi = \int_{-\infty}^{\infty} h(t - \xi - t_0) \varphi_p(\xi) \, d\xi
\]
\[
T_{t_0} \{ \mathcal{H} \{ \varphi_p(t) \} \} = \int_{-\infty}^{\infty} w(\xi, t - t_0) \varphi_p(\xi) \, d\xi = \int_{-\infty}^{\infty} h(t - t_0 - \xi) \varphi_p(\xi) \, d\xi
\]

Hence, we get here that
\[
\mathcal{H} \{ T_{t_0} \{ \varphi_p(t) \} \} = T_{t_0} \{ \mathcal{H} \{ \varphi_p(t) \} \}
\]
and therefore the system is **shift-invariant**.

**Question #5 (10 points) – The Discrete Fourier Transform**

Calculate the DFT (of order \( N \)) of the discrete one-dimensional signal with values defined for \( n = 0, \ldots, N - 1 \) via
\[
x_n = \begin{cases} 
1 & \text{for } n = 0, T, \ldots, (c - 1)T \\
0 & \text{otherwise}
\end{cases}
\]
where \( N = cT \) for some positive integer \( c \).

**Solution:**

Using the definition of Kronecker’s delta

\[
\delta_{n, n_0} = \begin{cases} 
1 & , \text{for } n = n_0 \\
0 & , \text{otherwise} 
\end{cases}
\]

we can write the signal as

\[
x_n = \sum_{l=0}^{c-1} \delta_{n, Tl}
\]

where \( T \) and \( c \) were defined in the question.

Recall the definition of the \( N^{th} \) order root of the unity: \( W_N = e^{\frac{i2\pi}{N}} \).

Then the \( k^{th} \) DFT coefficient is

\[
x_k^F = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} (W_N^*)^{kn} x_n = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} (W_N^*)^{kn} \sum_{l=0}^{c-1} \delta_{n, Tl} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \sum_{l=0}^{c-1} \delta_{n, Tl} \cdot (W_N^*)^{kn}
\]

Since \( N = cT \) we get that

\[
(W_N^*)^{kTl} = \left(e^{-\frac{i2\pi}{N}}\right)^{kTl} = e^{-\frac{i2\pi}{c}kl} = (W_c^*)^{kl}
\]

and therefore

\[
x_k^F = \frac{1}{\sqrt{N}} \sum_{l=0}^{c-1} (W_c^*)^{kl} = \frac{1}{\sqrt{N}} \sum_{l=0}^{c-1} (W_c^*)^{kl}
\]

For \( k = 0, c, ..., (T-1)c \) we get that

\[
\frac{1}{\sqrt{N}} \sum_{l=0}^{c-1} (W_c^*)^{kl} = \frac{1}{\sqrt{N}} \sum_{l=0}^{c-1} ((W_c^*)^c)^l = \frac{1}{\sqrt{N}} \sum_{l=0}^{c-1} (1)^l = \frac{1}{\sqrt{N}} c = \frac{1}{\sqrt{N}} N T = \frac{\sqrt{N}}{T}
\]

For \( k \neq 0, c, ..., (T-1)c \) we calculate the sum of the geometric series as

\[
\sum_{l=0}^{c-1} (W_c^*)^{kl} = \frac{1 - ((W_c^*)^c)^k}{1 - (W_c^*)^c} = \frac{1 - (W_c^*)^k}{1 - (W_c^*)^c} = \frac{1 - 1}{1 - (W_c^*)^c} = 0
\]

where we used the fact that \( (W_c^*)^c = \left(e^{-\frac{i2\pi}{c}n}\right)^c = e^{-i2\pi} = 1 \).

To conclude, we got that

\[
x_k^F = \begin{cases} 
\frac{\sqrt{N}}{T} & , \text{for } k = 0, c, ..., (T-1)c \\
0 & , \text{otherwise} 
\end{cases}
\]