1 LTI Systems and Signal Representations under Orthonormal Bases

1.1 The Dirac Delta Function and LTI Systems

We summarize the main points of the previous lecture

1. The Dirac delta function is a generalized function. Note that it is not a function according to the standard definition. It has the following properties.
   
   (a) Intuitively we can say that it satisfies
   \[
   \delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}.
   \]
   (1)

   (b) Its integral equals one:
   \[
   \int_{-\infty}^{\infty} \delta(\xi) d\xi = 1.
   \]
   (2)

   (c) For any continuous function \( f(t) \) and a constant \( t_0 \in \mathbb{R} \) we have
   \[
   \int_{-\infty}^{\infty} f(\xi) \delta(t - \xi) d\xi = f(t_0).
   \]
   (3)

2. We can represent any signal \( \varphi(t) \) using \( \delta(t) \) in a similar way to the representation of a vector
   \[
   \mathbf{a} = [a_1, a_2, \ldots, a_n]^T
   \]
   using the standard basis
   \[
   \mathbf{e}_i = \begin{cases} 0 & \text{for } i \neq \text{ith element} \\ 1 & \text{for } i \end{cases}, 
   \]
   \[
   \mathbf{e}_i = \begin{bmatrix} 0, \ldots, 0, 1, 0, \ldots, 0 \end{bmatrix}^T.
   \]
   As we can represent
   \[
   \mathbf{a} = \sum_{i=1}^{n} a_i \mathbf{e}_i,
   \]
   we can write the signal \( \varphi(t) \) as
   \[
   \varphi(t) = \int_{-\infty}^{\infty} \varphi(\xi) \delta(t - \xi) d\xi = \varphi(t) * \delta(t),
   \]
   (5)

   where \( \varphi(t) \) parallels the coefficients \( a_i \) and \( \delta(t - \xi) \) parallels the basis elements \( \mathbf{e}_i \) and instead of a discrete index \( i \) we have a continuous one \( \xi \).

3. Every LTI system can be represented as a convolution
   \[
   H_{LTI} \{ \varphi(t) \} = \int_{-\infty}^{\infty} \varphi(\xi) h_{LTI}(t - \xi) d\xi = \varphi(t) * h_{LTI}(t),
   \]
   where \( h_{LTI}(t) = H_{LTI} \{ \delta(t) \} \) is the impulse response of the system \( H_{LTI} \).
1.2 Orthonormal Bases with LTI Systems

Let \( \varphi(t) \) be a signal that has an approximate representation \( \hat{\varphi}(t) \) under a given family of orthonormal functions \( \{\psi_k(t)\}_{k=0}^{N-1} \), i.e,

\[
\varphi \equiv \hat{\varphi}(t) = \sum_{k=0}^{N-1} \varphi_k^* \psi_k(t).
\] (7)

We ask ourselves what is the output of a given system for such a signal. If the system \( H_{LT I} \) is LTI then we have

\[
H_{LT I} \{\psi_k(t)\} = \int_{-\infty}^{\infty} \psi_k(t) h_{LT I}(t-\xi) d\xi = \psi_k(t) * h_{LT I}(t).
\] (8)

We denote by \( h_k(t) \triangleq \psi_k(t) * h_{LT I}(t) \) the system's response to the basis function \( \psi_k(t) \). Since \( H_{LT I} \) is linear end up having

\[
H_{LT I} \{\hat{\varphi}(t)\} = \sum_{k=0}^{N-1} \varphi_k^* H_{LT I} \{\psi_k(t)\} = \sum_{i=0}^{N-1} \varphi_k^* h_k.
\] (9)

We see here that \( H_{LT I} \{\hat{\varphi}(t)\} \) is represented by the family of functions \( \{h_k(t)\}_{k=0}^{N-1} = \{\psi_k(t) * h_{LT I}(t)\}_{k=0}^{N-1} \).

Since \( \{\psi_k(t)\}_{k=0}^{N-1} \) is orthonormal, it is interesting to check what are the properties of the new family \( \{h_k(t)\}_{k=0}^{N-1} \).

We start by asking whether it is orthonormal, or what are the conditions for orthonormality. Let \( \psi_i(t), \psi_j(t) \in \{\psi_k(t)\}_{k=0}^{N-1} \). The inner product of their corresponding functions \( h_i(t) \) and \( h_j(t) \) is

\[
\langle h_i(t), h_j(t) \rangle = \int_{-\infty}^{\infty} h_i(\xi) \overline{h_j(\xi)} d\xi = \int_{-\infty}^{\infty} (\psi_i(\xi) * h_{LT I}(\xi)) \overline{(\psi_j(\xi) * h_{LT I}(\xi))} d\xi
\] (10)

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_i(\eta) h_{LT I}(\xi-\eta) d\eta \int_{-\infty}^{\infty} \psi_j(\xi) h_{LT I}(\xi-\eta) d\eta d\xi
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_i(\eta) \overline{\psi_j(\xi)} h_{LT I}(\xi-\eta) \overline{h_{LT I}(\xi-\eta)} d\eta d\xi d\eta.
\]

Notice that if it happens that \( \exists \lambda_{H}(i) \) such that

\[
h_i(t) = \int_{-\infty}^{\infty} \psi_i(\xi) h_{LT I}(t-\xi) d\xi = \lambda_{H}(i) \psi_j(t),
\] (11)

then we will have

\[
\langle h_i(t), h_j(t) \rangle = \int_{-\infty}^{\infty} \lambda_{H}(i) \psi_j(\xi) \overline{\lambda_{H}(l) \psi_j(\xi)} d\xi = \lambda_{H}(i) \overline{\lambda_{H}(l)} \int_{-\infty}^{\infty} \psi_j(\xi) \overline{\psi_i(\xi)} d\xi = \lambda_{H}(i) \overline{\lambda_{H}(l)} \int_{-\infty}^{\infty} \psi_j(\xi) \overline{\psi_j(\xi)} d\xi = \lambda_{H}(i) \overline{\lambda_{H}(l)} \int_{-\infty}^{\infty} \psi_j(\xi) \overline{\psi_j(\xi)} d\xi = \lambda_{H}(i) \overline{\lambda_{H}(l)} \left\{ 1 \quad \text{if} \quad i = l \right\}.
\] (12)

**Corollary 1.1.** Let \( H_{LT I} \) be a LTI system, \( \{\psi_k(t)\}_{k=0}^{N-1} \) be a family of orthonormal functions and \( \hat{\varphi}(t) = \sum_{k=0}^{N-1} \varphi_k^* \psi_k(t) \) be the (approximate) representation of a given signal \( \varphi(t) \) using this family. Assume that for all \( 0 \leq k \leq N-1 \) \( \exists \lambda_{H}(k) \) such that

\[
h_k(t) = H_{LT I} \{\psi_k(t)\} = \lambda_{H}(k) \psi_k(t)
\] (13)

then we have that

\[
H \{\hat{\varphi}(t)\} = \sum_{k=0}^{N-1} \varphi_k^* \lambda_{H}(k) \psi_k(t).
\] (14)
1.3 Eigenfunctions of LTI Systems

Notice that orthonormal functions that satisfies (13) are actually the eigenfunctions of the system. We seek for families of functions that will serve as eigenfunctions for LTI systems.

As a first try we look at \( \psi(t) = e^{\alpha t} \), where \( \alpha \in \mathbb{C} \) is a given constant. We have

\[
H_{LTI}\{\psi(t)\} = \int_{-\infty}^{\infty} e^{\alpha \xi} h_{LTI}(t-\xi) d\xi = \int_{-\infty}^{\infty} e^{\alpha(t-\xi)} h_{LTI}(\xi) d\xi = e^{\alpha t} \int_{-\infty}^{\infty} e^{-\alpha \xi} h_{LTI}(\xi) d\xi.
\]  

(15)

We get that \( e^{\alpha t} \) is an eigenfunction of any LTI system. However, the problem is that we cannot generate from \( e^{\alpha t} \) a family of orthonormal functions.

As a second try we look at \( \psi_k(t) = e^{j2\pi kt} \). In a similar way, we have

\[
H_{LTI}\{\psi_k(t)\} = \int_{-\infty}^{\infty} e^{j2\pi k\xi} h_{LTI}(t-\xi) d\xi = \int_{-\infty}^{\infty} e^{j2\pi k(t-\xi)} h_{LTI}(\xi) d\xi = e^{j2\pi kt} \int_{-\infty}^{\infty} e^{-j2\pi k\xi} h_{LTI}(\xi) d\xi.
\]

(16)

Thus,

\[
H_{LTI}\{\psi_k(t)\} = \lambda_{H}(k) e^{j2\pi kt}
\]

(17)

Note that in this case the functions \( \{\psi_k(t)\}_{k \in \mathbb{Z}} \) do form an orthonormal family and thus fulfill the condition we have looked for.

From this it follows that the functions \( \{\psi_k(t)\}_{k \in \mathbb{Z}} \) are eigenfunctions of any LTI system with eigenvalues equals to \( \lambda_{H}(k) = \mathcal{F}\{h_{LTI}(t)\}(k) \), i.e., the eigenvalues are the Fourier coefficients of the system impulse response.

We summarize this result in the following theorem

**Theorem 1.1.** Let \( H_{LTI} \) be a LTI system, then \( \forall \psi_k(t) = e^{j2\pi kt}, k \in \mathbb{Z} \)

\[
h_k(t) = H\{\psi_k(t)\} = \lambda_{H}(k)\psi_k(t),
\]

(18)

where

\[
\lambda_{H}(k) = \int_{-\infty}^{\infty} e^{-j2\pi k\xi} h_{LTI}(\xi) d\xi = \mathcal{F}\{h_{LTI}(t)\}(k),
\]

(19)

and \( h_{LTI} \) is the system’s impulse response.