Lecture 3
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1 Signal and Image Representation with Families of Orthonormal Functions

1.1 Solution of Bit Allocation Problem – Resolution-Quantization Trade-off

Given $B = N \cdot b$, the MSE of the signal representation is

$$E^2(N, b) = \int_0^1 \left( \phi(\xi) - \sum_{i=0}^{N-1} \phi_{opt, Q}^i \right)^2 d\xi$$

(1)

Using Taylor first order approximation we have

$$E^2(N, b) \approx \frac{2}{3} \cdot \frac{1}{N^2} \cdot Energy(\phi') + \frac{1}{12} \cdot \frac{(\phi_H - \phi_L)^2}{2^2b},$$

(2)

and this summarizes our discussion related to sampling and quantization under the assumption of selecting uniform intervals and by minimizing the MSE. We shall turn now to a more general discussion.

1.2 A General Theory for Signal (Image) Representation using a Family of Orthonormal Functions

Definition 1.1. Let $f, g \in C[0,1]$, where

$$C[0,1] = \{ f : [0,1] \rightarrow \mathbb{R} | f \text{ is continuous} \}.$$  

(3)

The inner product between two functions is denoted by $\langle \rangle$ and defined as

$$\langle f, g \rangle = \int_0^1 f(t) \cdot g(t) d\xi.$$  

(4)

Theorem 1.1. The function space $C[0,1]$ is an inner product space with respect to the known operations.

Definition 1.2. A family of orthonormal functions is defined as a set of $N$ functions $\{\psi_1(t), \psi_2(t), \ldots, \psi_N(t)\}$, where $\psi_i : [0,1] \rightarrow \mathbb{R}$ for every $1 \leq i \leq N$ and

$$\langle \psi_i, \psi_k \rangle = \delta_{i,k}$$

(5)

for every $1 \leq i, k \leq N$, where $\delta_{i,k} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$ is the Kronecker delta function.
1.2.1 Examples for Orthonormal Families

There are many examples for families of orthonormal functions. For example, we have

1. The family

\[ \{ \psi_i \}_{i=1,2,\ldots,N} \]

where \( \psi_i = \sqrt{N} \cdot 1_{\Delta_{i,1}}(t) \) for every \( 1 \leq i \leq N \).

2. Sines and Cosines base families.

3. The Haar family.

4. The Walsh-Hadamard family.

5. LETS: Wavelets/ Curvelets.

Another example we can find in Fig. 1.
1.2.2 Signal Representation using an Orthonormal Family

Given an orthonormal family of functions and a signal \( \varphi(t) \) (or an image \( I(x, y) \)), we want to represent (to approximate) the signal \( \varphi \) using the set \( \{ \psi_i | i \in \{1, 2, \ldots, N\} \} \) in the following way:

\[
\varphi(t) \cong \sum_{i=1}^{N} \varphi_i \cdot \psi_i(t) \overset{\text{approx}}{=} \hat{\varphi}(t),
\]

where \( \varphi_i \) are real coefficients that are a result of sampling with or without quantization.

1.2.3 Optimal Representation Coefficients Selection for a Given Orthonormal Family

We shall choose \( \varphi_1, \varphi_2, \ldots, \varphi_N \) that will give us the best approximation for \( \varphi(t) \) in the MSE sense. Defining

\[
\Sigma(t) \overset{\Delta}{=} \varphi(t) - \hat{\varphi}(t) = \varphi(t) - \sum_{i=1}^{N} \varphi_i \cdot \psi_i(t),
\]

we aim at minimizing

\[
\mathbb{E}^2(\varphi_1, \varphi_2, \ldots, \varphi_N) = \int_{0}^{1} \Sigma^2(t) dt = \int_{0}^{1} \left[ \varphi(t) - \sum_{i=0}^{N} \varphi_i \cdot \psi_i(t) \right]^2 dt.
\]

Therefore, the optimal coefficients for a given orthogonal family will be

\[
\{ \varphi_1^{opt}, \varphi_2^{opt}, \ldots, \varphi_N^{opt} \} = \arg \min_{\varphi_1, \varphi_2, \ldots, \varphi_N} \mathbb{E}^2(\varphi_1, \varphi_2, \ldots, \varphi_N).
\]

For finding the optimal coefficients we require the partial derivatives with respect to each of them to be equal to zero:

\[
\frac{\partial}{\partial \varphi_j} \mathbb{E}^2(\varphi_1, \varphi_2, \ldots, \varphi_N) = 0
\]

Thus, we have that for each \( 1 \leq i \leq N \)

\[
\frac{\partial}{\partial \varphi_j} \left[ \int_{0}^{1} \left( \varphi(t) - \sum_{i=1}^{N} \varphi_i \cdot \psi_i(t) \right)^2 dt \right] = \frac{\partial}{\partial \varphi_i} \left[ \int_{0}^{1} \left( \varphi^2(t) - 2\varphi(t) \cdot \sum_{i=1}^{N} \varphi_i \cdot \psi_i(t) \right) \right] \]

\[
= \frac{\partial}{\partial \varphi_j} \left[ \int_{0}^{1} \varphi^2(t) dt - 2 \int_{0}^{1} \varphi(t) \cdot \sum_{i=1}^{N} \varphi_i \cdot \psi_i(t) dt + \int_{0}^{1} \left( \sum_{i=1}^{N} \varphi_i \cdot \psi_i(t) \right)^2 dt \right]
\]

\[
= \frac{\partial}{\partial \varphi_j} \left[ \int_{0}^{1} \varphi^2(t) dt - 2 \sum_{i=1}^{N} \varphi_i \int_{0}^{1} \varphi(t) \cdot \psi_i(t) dt + \sum_{i=1}^{N} \varphi_i^2 \cdot \psi_i^2(t) dt \right]
\]

\[
= -2 \int_{0}^{1} \varphi(t) \cdot \psi_j(t) dt + 2 \varphi_j \int_{0}^{1} \psi_j^2(t) dt + \sum_{i \neq j} \varphi_i \int_{0}^{1} \psi_i(t) \psi_j(t) dt = 0.
\]

Since \( \{ \psi_i | i \in \{1, 2, \ldots, N\} \} \) is an orthonormal family we have that

\[
\int_{0}^{1} \psi_i(t) \psi_j(t) dt = 0 \quad \text{for every } i \neq j
\]

and

\[
\int_{0}^{1} \psi_j^2(t) dt = 1 \quad \text{for every } 1 \leq j \leq N
\]
Plugging (14) and (15) in (13) provides us with an equation for the optimal coefficient value:

\[
-2 \int_0^1 \varphi(t) \cdot \psi_j(t) \, dt + 2\varphi_j = 0.
\]

(16)

Thus, we have

\[
\varphi_j^{opt} = \int_0^1 \varphi(t) \cdot \psi_j(t) \, dt.
\]

(17)

### 1.2.4 Optimal Representation Error for a Given Orthonormal Family

Having the optimal coefficients

\[
\varphi_i^{opt} = \int_0^1 \varphi(t) \cdot \psi_i(t) \, dt \quad 1 \leq i \leq N,
\]

we calculate the representation error.

\[
\mathbb{E}^2(\varphi_i^{opt}) = \int_0^1 \varphi^2(t) \, dt - 2\int_0^1 \left(\varphi(t) \cdot \sum_{i=1}^N \varphi_i^{opt} \cdot \psi_i(t)\right) \, dt + \int_0^1 \left(\sum_{i=1}^N \varphi_i^{opt} \cdot \psi_i(t)\right)^2 \, dt
\]

(19)

\[
= \int_0^1 \varphi^2(t) \, dt - 2 \sum_{i=1}^N \varphi_i^{opt} \int_0^1 \varphi(t) \cdot \psi_i(t) \, dt + \int_0^1 \left(\sum_{i=1}^N \varphi_i^{opt} \cdot \psi_i(t)\right)^2 \, dt
\]

\[
= \int_0^1 \varphi^2(t) \, dt - 2 \sum_{i=1}^N \left(\varphi_i^{opt}\right)^2 + \int_0^1 \sum_{i=1}^N \left(\varphi_i^{opt}\right)^2 \cdot \psi_i^2(t) \, dt + \sum_{i \neq k} \varphi_i^{opt} \varphi_k^{opt} \int_0^1 \psi_i(t) \psi_k(t) \, dt
\]

\[
= \int_0^1 \varphi^2(t) \, dt - \sum_{i=1}^N \left(\varphi_i^{opt}\right)^2,
\]

where the last equality is due to (14) and (15). Using the definition of \( \varphi_i^{opt} \) in (17) the error’s final form is:

\[
\mathbb{E}^2(\varphi_i^{opt}) = \int_0^1 \varphi^2(t) \, dt - \sum_{i=1}^N \left(\varphi_i^{opt}\right)^2 = \int_0^1 \varphi^2(t) \, dt - \sum_{i=1}^N \left(\int_0^1 \varphi(t) \cdot \psi_i(t) \, dt\right)^2
\]

(20)

### 1.2.5 Sampling and Quantization Error for a Given Orthonormal Family

Having calculated the sampling error, we turn to ask ourselves what is contribution of the quantization to the representation error. Denoting by \( \varphi_i^{opt,Q} \) the quantized version of \( \varphi_i^{opt} \), we have that the quantized representation is given by

\[
\varphi(t) \equiv \sum_{i=1}^N \varphi_i^{opt,Q} \cdot \psi_i(t) \triangleq \varphi^Q(t),
\]

(21)

and its error is

\[
\mathbb{E}^2(\varphi_i^{opt,Q}) = \int_0^1 \left(\varphi(\xi) - \varphi^Q(\xi)\right)^2 \, d\xi
\]

(22)

\[
= \int_0^1 \varphi^2(\xi) \, d\xi - 2 \sum_{i=1}^N \left(\varphi_i^{opt,Q} \cdot \int_0^1 \varphi(\xi) \cdot \psi_i(\xi) \, d\xi\right) + \int_0^1 \left(\sum_{i=1}^N \varphi_i^{opt,Q} \cdot \psi_i(\xi)\right)^2 \, d\xi
\]

\[
= \int_0^1 \varphi^2(\xi) \, d\xi - 2 \sum_{i=1}^N \left(\varphi_i^{opt,Q} \cdot \varphi_i^{opt}\right) + \sum_{i=1}^N \left(\varphi_i^{opt,Q}\right)^2,
\]
where in the last equality the second term follows from the fact that $\varphi_{i}^{\text{opt}} = \int_{0}^{1} \varphi(\xi) \cdot \psi_{i}(\xi) d\xi$ and the third term is derived using the same steps used for the derivation of the sampling error. By adding and subtracting $\sum_{i=1}^{N} (\varphi_{i}^{\text{opt}})^2$ we have

$$E^2(\varphi_{i}^{\text{opt},Q}) = \int_{0}^{1} \varphi^2(\xi) d\xi - \sum_{i=1}^{N} \left((\varphi_{i}^{\text{opt},Q})^2 - 2\varphi_{i}^{\text{opt},Q} \cdot \varphi_{i}^{\text{opt}} + (\varphi_{i}^{\text{opt}})^2\right) - \sum_{i=1}^{N} (\varphi_{i}^{\text{opt}})^2 \tag{23}$$

Thus, we end up having

$$E^2\left(\varphi_{i}^{\text{opt},Q}\right) = E^2\left(\varphi_{i}^{\text{opt}}\right) + \sum_{i=1}^{N} \left(\varphi_{i}^{\text{opt},Q} - \varphi_{i}^{\text{opt}}\right)^2, \tag{24}$$

where $E^2(\varphi_{i}^{\text{opt}})$ is the sampling error with no quantization and $\sum_{i=1}^{N} \left(\varphi_{i}^{\text{opt},Q} - \varphi_{i}^{\text{opt}}\right)^2$ is the error due to the quantization. As in the previous lecture, we have here a tradeoff between the number of functions in the orthonormal family and the number of bits used for coding the each coefficient. If we use $b$ bits for coding the coefficient then the average error of the coefficients that fall in the quantization interval $r_i$ is approximately (based on similar considerations as we had in the last lecture)

$$\frac{1}{12} \frac{r_i^2}{2^b}. \tag{25}$$

Here we have a more complicated bit allocation problem from the one we had last lecture since we need to select the size of the quantization intervals. This can be done by drawing the error as function of the number of bit values, where for each number of bits the Max-Lloyd criterion should be used to determine the sizes of $r_i$. 