Signal and Image Processing (236327)

Tutorial 5

Signal Representation Over Orthonormal Bases
Reminder:
A Family of Orthonormal Functions

A set of orthonormal (continuous and defined over \([0,1]\)) functions

\[
\{\psi_1(t), \psi_2(t), \ldots, \psi_N(t), \ldots\}
\]

holds the **orthonormality condition**

\[
\langle \psi_i(t), \psi_j(t) \rangle = \begin{cases} 
1 & \text{for } i = j \\
0 & \text{for } i \neq j
\end{cases}
\]

where the inner-product is defined as

\[
\langle \psi_i, \psi_j \rangle = \int_0^1 \psi_i(t)\psi_j(t)dt
\]
Reminder:
Representing Signals using Orthonormal Functions

We represent a given signal $\varphi(t)$ by approximating it using a set of $N$ orthonormal functions:

$$\hat{\varphi}(t) = \sum_{i=1}^{N} \varphi_i \cdot \psi_i(t)$$

where $\{\varphi_i\}_{i=1}^{N}$ are the representation coefficients.

What are the optimal representation-coefficients?
Reminder:
Representing Signals using Orthonormal Functions

What are the optimal representation-coefficients?

The MSE for $N$-term approximation is

$$\text{MSE}(N) = \int_0^1 (\varphi(t) - \hat{\varphi}(t))^2 dt = \int_0^1 \left( \varphi(t) - \sum_{i=1}^{N} \varphi_i \cdot \psi_i(t) \right)^2 dt$$

The optimal $\{\varphi_i\}_{i=1}^{N}$ are given by demanding

$$\frac{\partial \text{MSE}(N)}{\partial \varphi_i} = 0 \quad \text{for} \quad i = 1, \ldots, N$$

that results in $\varphi_i^{opt} = \langle \varphi(t), \psi_i(t) \rangle = \int_0^1 \varphi(t)\psi_i(t)dt$
Reminder:
Representing Signals using Orthonormal Functions

We can further develop the MSE for $N$-term approximation:

$$MSE(N) = \int_{0}^{1} \left( \varphi(t) - \sum_{i=1}^{N} \varphi_{i}^{\text{opt}} \cdot \psi_{i}(t) \right)^{2} dt$$

$$= \int_{0}^{1} \left[ \varphi^{2}(t) - 2 \varphi(t) \cdot \sum_{i=1}^{N} \varphi_{i}^{\text{opt}} \cdot \psi_{i}(t) + \left( \sum_{i=1}^{N} \varphi_{i}^{\text{opt}} \cdot \psi_{i}(t) \right)^{2} \right] dt$$

$$= \int_{0}^{1} \varphi^{2}(t) dt - \int_{0}^{1} 2 \varphi(t) \cdot \sum_{i=1}^{N} \varphi_{i}^{\text{opt}} \cdot \psi_{i}(t) dt + \int_{0}^{1} \left( \sum_{i=1}^{N} \varphi_{i}^{\text{opt}} \cdot \psi_{i}(t) \right)^{2} dt$$

$$= \int_{0}^{1} \varphi^{2}(t) dt - 2 \sum_{i=1}^{N} \left( \varphi_{i}^{\text{opt}} \right)^{2} + \sum_{i=1}^{N} \left( \varphi_{i}^{\text{opt}} \right)^{2}$$

$$= \int_{0}^{1} \varphi^{2}(t) dt - \sum_{i=1}^{N} \left( \varphi_{i}^{\text{opt}} \right)^{2}$$
The Perspective of Generalized Sampling

Standard Sampling

Let us consider the standard sampling functions:

\[ \psi^s_i(t) = \begin{cases} \sqrt{N} & \text{for } t \in \left[ \frac{i-1}{N}, \frac{i}{N} \right] \\ 0 & \text{otherwise} \end{cases} \]

Properties of these functions:

• **Tiles the entire domain** [0,1].
• Local non-overlapping supports.
• **Constant valued** over the supports.
The Perspective of Generalized Sampling

Standard Sampling

Show orthonormality:

For \( i \neq j \):

\[
\langle \psi_i^S(t), \psi_j^S(t) \rangle = \int_0^1 \psi_i^S(t)\psi_j^S(t) dt
\]

\[
= \int_{i-1}^{N_i} \sqrt{N} \cdot 0 dt + \int_{N_j}^{j-1} 0 \cdot \sqrt{N} dt = 0
\]

For \( i = j \):

\[
\langle \psi_i^S(t), \psi_i^S(t) \rangle = \int_0^1 \psi_i^S(t)\psi_i^S(t) dt = \int_{i-1}^{N_i} \sqrt{N} \cdot \sqrt{N} dt = 1
\]
The Perspective of Generalized Sampling

Standard Sampling

The general formulae reduce to the previous results of the standard case:

- For example, the \textbf{optimal coefficients} are the normalized averages over the sampling intervals:

\[
\varphi_{i}^{\text{opt}} = \langle \varphi(t), \psi_i(t) \rangle = \int_0^1 \varphi(t)\psi_i(t)dt = \int_{i-1}^{i} \sqrt{N} \cdot \varphi(t)dt
\]

\[
= \frac{1}{\sqrt{N}} \left( \frac{1}{1/N} \int_{i-1}^{i} \varphi(t)dt \right)
\]

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Standard Sampling: The Implications of Locality

The standard-basis functions have **local non-overlapping** supports, therefore, a **subset of the coefficients** allows reconstruction only over parts of the signal’s domain:

The reconstructed signal is not defined over the intervals corresponding to the omitted coefficients.  

No multi-resolution support.
Unitary Matrices

The orthonormality condition for a square matrix $U$ of size $N \times N$:

- For general complex-valued matrices: $U^* U = I$
  - where $*$ is the conjugate-transpose operator.
- For real-valued matrices: $U^T U = I$
  - where the conjugate-transpose reduces to transpose only.

The above implies that the matrix columns/rows are orthonormal:

$$u_i^* u_j = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

where $u_i$ is the $i^{th}$ column (or row) of the matrix $U$.

I.e., $\{u_i\}_{i=1}^N$ are $N$ linearly-independant vectors of length $N$.
Therefore, they form a basis of $\mathbb{R}^N$ (in the real case) or $\mathbb{C}^N$ (in the complex case).
Constructing Orthonormal Functions from a Unitary Matrix

Given a unitary matrix $U$ of size $N \times N$, an orthonormal set of $N$ continuous functions can be constructed:

$$\psi_i(t) = \sum_{k=1}^{N} u_{k,i} \cdot \psi^s_k(t) \quad , i = 1, \ldots, N$$

where,

$u_{k,i}$ is the matrix-element placed in the $i^{th}$ column in the $k^{th}$ row.

$\psi^s_k(t)$ is the $k^{th}$ function of the standard family (for $N$ samples).

The orthonormality of the columns, $\{u_i\}_{i=1}^{N}$, leads to orthonormality of the functions $\{\psi_i(t)\}_{i=1}^{N}$.

(Proof was given in the lecture).
The Hadamard Basis

The Hadamard matrices are recursively defined as follows:

\[ H_1 = 1 \]

\[ H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \]

\[ H_{2^n} = H_2 \otimes H_{2^{n-1}} = \frac{1}{\sqrt{2}} \begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{bmatrix} \quad \text{for } n = 2, 3, \ldots \]

Note that \( H_{2^n} \) is a \( 2^n \times 2^n \) matrix.

Examples:

\[ H_4 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \]

\[ H_8 = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \]
The Walsh-Hadamard Basis

**Walsh-Hadamard** matrices are the Hadamard matrices with rows **ordered according to their sequency** (the number of sign-changes).

**Example:**

\[
H_4 = \frac{1}{\sqrt{4}} \begin{bmatrix} 
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 \\
\end{bmatrix}
\]

\[
W_4 = \frac{1}{\sqrt{4}} \begin{bmatrix} 
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 \\
\end{bmatrix}
\]

The 4 **orthonormal continuous functions** corresponding to the above Walsh-Hadamard matrix \(W_4\) are:

\[
\psi_i(t) = \sum_{k=1}^{4} w_{k,i} \cdot \psi^s_k(t) \quad , i = 1, \ldots, 4
\]

where \(\psi^s_k(t)\) are the standard sampling functions, and \(w_{k,i}\) is the \((k, i)\) element of the matrix \(W_4\).
Exercise: Comparing Sampling using the Standard and Hadamard Functions

The signal $\varphi(t) = \begin{cases} t, & \text{for } t \in [0,1] \\ 0, & \text{otherwise} \end{cases}$

is sampled using the standard sampling function to have $N=4$ samples.

Calculate the representation-coefficients and demonstrate the N-term approximation and its error.
Exercise: Comparing Sampling using the Standard and Hadamard Functions

Consider the standard sampling functions ($N=4$).

What are the optimal coefficients?

Let’s define the sampling-interval size $\Delta = \frac{1}{N} = \frac{1}{4}$.

The coefficients are the normalized interval averages which in this case (where $\varphi(t)$ is linear) are the normalized interval centers:

$$\varphi_i^s = \frac{1}{\sqrt{N}} \left( \frac{1}{1/N} \int_{i-1/N}^{i/N} \varphi(t) dt \right) = \frac{1}{\sqrt{N}} \Delta \cdot \left( i - \frac{1}{2} \right)$$

$$\varphi_1^s = \frac{1}{16} \quad \varphi_2^s = \frac{3}{16} \quad \varphi_3^s = \frac{5}{16} \quad \varphi_4^s = \frac{7}{16}$$
Exercise: Comparing Sampling using the Standard and Hadamard Functions

The K-term approximation ($K = 1, \ldots, N$) for the standard sampling functions ($N=4$):

1 term
\[ \varphi_s^1 = \frac{7}{16} \]

2 terms
\[ \varphi_s^2 = \frac{5}{16} \]

3 terms
\[ \varphi_s^3 = \frac{3}{16} \]

4 terms
\[ \varphi_s^4 = \frac{1}{16} \]

\[
MSE(1) = \frac{1}{3} \left( \frac{7}{16} \right)^2 = 0.1419 \\
MSE(2) = 0.0443 \\
MSE(3) = 0.0091 \\
MSE(4) = 0.0052
\]
Exercise: Comparing Sampling using the Standard and Hadamard Functions

Now we represent the signal using the Walsh-Hadamard function ($N = 4$). What are the coefficients and the MSE?

For $i = 1, ..., 4$: $\varphi^w_i = \langle \varphi(t), \psi^w_i(t) \rangle = \int_0^1 t \psi^w_i(t) dt$

We will use the following auxiliary-integral: $I(a, b) = \int_a^b t dt = \frac{t^2}{2} \bigg|_a^b = \frac{b^2 - a^2}{2}$

\[
\varphi^w_1 = \langle \varphi(t), \psi^w_1(t) \rangle = \int_0^1 t dt = I(0, 1) = \frac{1}{2}
\]

\[
\varphi^w_2 = \langle \varphi(t), \psi^w_2(t) \rangle = I \left( 0, \frac{1}{2} \right) - I \left( \frac{1}{2}, 1 \right) = \frac{1}{8} - \frac{3}{8} = -\frac{1}{4}
\]
Exercise: Comparing Sampling using the Standard and Hadamard Functions

\[ \varphi_3^w = \langle \varphi(t), \psi_3^w(t) \rangle = I\left(0, \frac{1}{4}\right) - I\left(\frac{1}{4}, \frac{3}{4}\right) + I\left(\frac{3}{4}, 1\right) \]

\[= \frac{1}{32} - \frac{8}{32} + \frac{7}{32} = 0 \]

\[ \varphi_4^w = \langle \varphi(t), \psi_4^w(t) \rangle = I\left(0, \frac{1}{4}\right) - I\left(\frac{1}{4}, \frac{1}{2}\right) + I\left(\frac{1}{2}, \frac{3}{4}\right) - I\left(\frac{3}{4}, 1\right) \]

\[= \frac{1}{32} - \frac{3}{32} + \frac{5}{32} - \frac{7}{32} = -\frac{1}{8} \]
Exercise: Comparing Sampling using the Standard and Hadamard Functions

The K-term approximation for the **Walsh-Hadamard** sampling functions (N=4):

1 term
\[ \varphi_1^w = \frac{1}{2} \]

2 terms
\[ \varphi_2^w = -\frac{1}{4} \]

3 terms
\[ \varphi_3^w = 0 \]

4 terms
\[ \varphi_4^w = -\frac{1}{8} \]

The 4-term approximation equals to the 3-term one.

\[ MSE(4) = MSE(3) \]

\[ MSE(1) = \frac{1}{3} - \left( \frac{1}{2} \right)^2 = 0.0833 \]

\[ MSE(2) = 0.0208 \]

\[ MSE(3) = 0.0052 \]

\[ MSE(K) = \int_0^1 \varphi^2(t)dt - \sum_{i=1}^{K} (\varphi_i)^2 \]
Exercise: Comparing Sampling using the Standard and Hadamard Functions

Plotting the error-curves shows the superiority of approximation using the Walsh-Hadamard functions:

- Note that in both bases the error converges to the same value (why?).