Partial Solution for Homework #4

Solution for Question #1

a. For the unitary transform $U$ we define the transformed signal as $y = Ux$.
The autocorrelation matrix of the vector $y$ is
\[ R_y = E[yy^T] = E[Ux(Ux)^T] = E[UU^T] = UR_xU^T \]

b. The measurement vector $y$ (of size $n \times 1$) is given, we denote the linear filter as the $N \times N$ matrix $H$, and the filtered signal is $\hat{x} = Hy$.
The estimation (or filtering) error is defined as $e = x - \hat{x}$, where $x$ is the signal that we try to recover from $y$.
The expected MSE can be written as:
\[ E[e^2] = E[(x - \hat{x})^T(x - \hat{x})] = E[(x - Hy)^T(x - Hy)] \]
The demand for optimality of the filtering in sense of minimal expected MSE can be expressed by:
\[ \frac{d}{dH} E[(x - Hy)^T(x - Hy)] = 0 \]
Developing this derivative expression yields
\[ \frac{d}{dH} E[(x - Hy)^T(x - Hy)] = E \left[ \frac{d}{dH} \{(x - Hy)^T(x - Hy)\} \left\{ \frac{d}{dH}(x - Hy) \right\} \right] \]
\[ = -2E[y^T(x - Hy)] \]
\[ = -2E[y^T e] \]

Hence, we $E[y^T e] = 0$ is held as requested.
Solution for Question #2

Let us consider the case where the given signal has uncorrelated components with identical second order statistics. In this case the autocorrelation matrix is of the form $R_x = \sigma^2 I$, where $\sigma^2$ is the variance of each of the samples.

In this case, any unitary transform will result in a signal $y$ that has the same autocorrelation as $x$:

$$R_y = U^* R_x U = U^* \sigma^2 I U = \sigma^2 U^* U = \sigma^2 I = R_x$$

Hence, any unitary transform is equally optimal to the KLT.

Now let’s consider another case where the given signal has uncorrelated components with different second-order statistics (and specifically different variances), i.e., $R_x$ is a general diagonal matrix with diagonal-values that are not necessarily equal.

In this case an arbitrary unitary transform is not necessarily equally optimal to the KLT as the autocorrelation matrix of the transformed signal is not necessarily a diagonal matrix. The following example illustrates this:

The two component vector $x$ has uncorrelated components that and the following autocorrelation matrix: $R_x = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

Of course that the 2x2 identity matrix is a KLT matrix for this signal.

Let us consider the unitary transform $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ that leads to the transformed signal $y$ that has the autocorrelation matrix

$$R_y = U^* R_x U = \begin{pmatrix} 3 & 1 \\ 2 & 2 \\ 2 & 3 \\ \frac{1}{2} & 2 \end{pmatrix}$$

which is not a diagonal matrix, and therefore sub-optimal and not equal to the KLT.
Solution for Question #3

After mathematically proving that $\text{Trace} [R_x] = \text{Trace} [R_y]$ for unitary transformation, it is important to understand its corresponding signal-processing interpretation as the famous property of energy preservation under unitary transform.

General Remarks to the Matlab Part

The experiment in this part is intended to exemplify the following theoretical concepts that were taught in class:

1. KLT is the optimal transform in terms of minimal expected MSE for K-term approximation of signals (i.e., the energy compaction is optimal). This means that no other transform can get lower average MSE values in this experiment, this is true due to:
   a. You transform a large number of vectors so the empirical average of the approximation error represents well the probabilistic expectation (which is part of the “optimality sense” of the KLT).
   b. The first order Markov approximation is good enough here, as the peppers.png image has a very high horizontal correlation coefficient, which also represents quite well the entire image (i.e., there are no image areas with different texture-levels or content-complexities).
2. DCT is a good approximation for the KLT of first-order Markov models, and indeed the average MSE values of its results should be close or similar to those of the KLT, but not lower!
3. The DFT performs significantly worse than the KLT and DCT, so its average MSE is higher.