Complexity of algebraic computation

Homework assignment #10

1. Let

\[ F(x_1, x_2, \ldots, x_n) = \left\{ \sum_{i=1}^{n} a_{ij} x_i : j = 1, 2, \ldots, m \right\} \]

be a set of homogeneous linear forms over an infinite field \( K \) and let \( \mathcal{A} \) be a straight-line algorithm over \( K \) that computes \( F(x_1, x_2, \ldots, x_n) \) in \( a \) additions and \( m \) multiplications/divisions. Prove that there is a straight-line algorithm \( \mathcal{A}' \) without multiplications and divisions that computes \( F(x_1, x_2, \ldots, x_n) \) over \( K \) in \( a + m \) additions.

2. Define the notion of quadratic complexity \( \tau \) of a set of quadratic forms and prove that for a set of bilinear forms \( B \), \( \tau(B) \leq 2 \beta(B) \).

3. Prove that the sets bilinear forms dual to the bilinear forms defined by the product of two complex numbers (over \( \mathbb{R} \)) are also defined by the product of two complex numbers.

4. Let \( M(m, n, \ell) \) denote the (minimal) number of multiplications need for computing the product of an \( m \times n \) matrix by an \( n \times \ell \) matrix by a bilinear algorithm. Prove that

\[ M(m, n, \ell) = M(n, \ell, m) = M(\ell, m, n). \]

5. Draw the graphs of the linear algorithms for the 4- and 8-point finite Fourier transforms. What is the relationship between these graphs?

6. (a) What is the value of \( \mu_{F_2}(3) \)?

(b) What is the value of \( \mu_{F_1}(4) \)?

7. Where in the proof of the lower bound on the bilinear complexity of polynomial multiplication was used the assumption that the algorithms are bilinear?
**Hint** for questions 3 and 4 obtain the dual sets of the set of bilinear forms

\[ \{ z_k = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ijk}x_y y_j : k = 1, 2, \ldots, \ell \} \]

from the trilinear form

\[ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{\ell} a_{ijk}x_y y_j z_k. \]