Transformations

\[
\begin{bmatrix}
\cos 90^\circ & \sin 90^\circ \\
-sin 90^\circ & \cos 90^\circ \\
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
\end{bmatrix} = \begin{array}{c}
\bigcirc \\
\bigcirc \\
\end{array}
\]
Geometric Transformations

- Transformation $\tau$ is one-to-one and onto mapping of $\mathbb{R}^n$ to itself (in general)

- Affine transformation $\tau(V) = AV + b$
  - $A$ – matrix
  - $b$ – scalar

- In CG Geometric Transformations are, for the most part, affine transformations with geometric meaning

- Note: transformation has meaning only for vectors $\Rightarrow$ for point $P$, $\tau(P)$ is a transformation of vector from the origin to $P$
Transformations

- Transforming an object = transforming all its points

- For a polygonal model = transforming its vertices
  - Is it true for any transformation?
  - Why is it true for us?
Applications

- Viewing
- Modeling
- Animation
- Image manipulation
Scaling

- $V = (v_x, v_y)$ – vector in $XY$ plane

- **Scaling** operator $S$ with parameters $(s_x, s_y)$:

$$S(s_x, s_y)(V) = (v_x s_x, v_y s_y)$$
Scaling

- Matrix form:

\[
S^{(s_x, s_y)}(V) = \begin{pmatrix} v_x \\ v_y \end{pmatrix} \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} = \begin{pmatrix} v_x s_x \\ v_y s_y \end{pmatrix}
\]

- Independent in \(x\) and \(y\)
  - Non uniform scaling is allowed

- What is the meaning of scaling by zero?
¶ Polar form:

\[ V = (v_x, v_y) = (r \cos \alpha, r \sin \alpha) \]

¶ Rotating \( V \) anti-clockwise by \( \theta \) to \( W \):
Rotation

- Matrix form:

- \( \text{Rotation operator } R \) with parameter \( \theta \) at the origin:
Rotation Properties

- $R^\theta$ is orthogonal

\[ (R^\theta)^{-1} = (R^\theta)^T \]

- $R^{-\theta}$ - rotation by $-\theta$ is

\[
R^{-\theta}(V) = (v_x, v_y) \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} = (R^\theta)^{-1}
\]
Translation

- Translation operator $T$ with parameters $(t_x, t_y)$:

$$T^{t_x,t_y}(V) = (v_x + t_x, v_y + t_y)$$

- Can you express $T$ in a matrix form?
Translation - Homogeneous Coordinates

- To represent $T$ in matrix form
  - introduce homogeneous coordinates:

- Conversion (projection) from homogeneous space to Euclidean:

- In homogeneous coordinates:
  
  \[(2, 2, 1) = (4, 4, 2) = (1, 1, 0.5)\]
Translation

- Using homogeneous coordinates, the translation operator may be expressed as:
Matrix Form

Why did we insist on using a **Matrix** form to represent the **transformations**?

Hint: How would you go about transforming an object of *m* vertices through a sequence of *n* transformations?
Transformation Composition

What operation rotates $XY$ by $\theta$ around $P = (p_x, p_y)$?

Answer:
- Translate $P$ to origin
- Rotate around origin by $\theta$
- Translate back
Transformation Composition
Transformations Quiz

- What do these transformations do?

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 0 \\
0 & -1 \\
-1 & 0 \\
0 & 1 \\
1 & 0 \\
1 & 1 \\
1 & a \\
0 & 1 \\
\end{bmatrix}
\]

- And these homogeneous ones?

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1 \\
0 & 0 & 0.5 \\
\end{bmatrix}
\]

- How can one mirror a planar object through an arbitrary line in $XY$?

- Can one rotate a planar object in the $XY$ plane via mirroring?
Rotate by Shear

- Shear
  
  \[
  \begin{bmatrix}
  1 & 0 \\
  1 & 1
  \end{bmatrix}
  \begin{bmatrix}
  1 & 1 \\
  0 & 1
  \end{bmatrix}
  \begin{bmatrix}
  1 & \alpha \\
  0 & 1
  \end{bmatrix}
  \begin{bmatrix}
  1 & 0 \\
  \alpha & 1
  \end{bmatrix}
  \]

- Rotation by \(0 \leq \theta \leq \pi/2 \equiv \) composition of 3 shears
Rotate by Shear (cont.)

- Solve for $x$, $y$, $z$:

$$\cos \theta = 1 + xy = yz + 1$$
$$\sin \theta = -y = z + xyz + x$$

- Solution:

$$y = -\sin \theta$$
$$x = z = -\tan(\theta/2)$$

**Question**: When is rotation-using-three-shears useful?

**Question**: Can we derive a rotation via **two** shears?
Rotate by Shear (cont.)

- What does the composition of the following two scaled shears do?

\[
\begin{pmatrix}
  1 & \tan \theta \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
\end{pmatrix}
\]

- Why use it?

- Is it convenient?

- What happens when \( \theta \to \pi/2 \)?
Rotation Approximation

- For small angles can approximate:
  \[ x' = x \cos \theta - y \sin \theta \approx x - y \theta \]
  \[ y' = x \sin \theta + y \cos \theta \approx x \theta + y \]

  Since \( \cos \theta \to 1 \) and \( \sin \theta \to \theta \) as \( \theta \to 0 \) (Taylor expansion of \( \sin/\cos \))

- Example (steps of \( \theta \)):

- When is this rotation approximation useful?
Rotation Approximation

- App. rotation matrix

\[ AR^\theta = \begin{bmatrix} 1 & \theta \\ -\theta & 1 \end{bmatrix} \]

- Matrix has to be orthogonal & normalized
- \( AR^\theta \) is orthogonal
- \( AR^\theta \) not normalized
- Normalize
3D Transformations

- All 2D transformations extend to 3D
- In homogeneous coordinates:

<table>
<thead>
<tr>
<th>Scale</th>
<th>Translate</th>
<th>Rotate (around $z$ axis)</th>
</tr>
</thead>
</table>

**Question:** What is ? ? ?
3D Transformations

- Questions (commutativity):
  - Scaling: $S_1S_2 = S_2S_1$?
  - Translation: $T_1T_2 = T_2T_1$?
  - Rotation: $R_1R_2 = R_2R_1$?
Example: Arbitrary Rotation

Problem:
Given two orthonormal coordinate systems $XYZ$ and $UVW$, find a transformation from one to the other.
Arbitrary Rotation

- **Answer:**
  - Transformation matrix $R$ whose rows are $U$, $V$, $W$:

- **Proof:**

  $$R(X) = (1,0,0) \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = (u_x, u_y, u_z) = U$$

- Similarly $R(Y) = V$ & $R(Z) = W$
Arbitrary Rotation (cont.)

- Inverse (=transpose) transformation, $R^{-1}$, provides mapping from $UVW$ to $XYZ$

- E.g.

$$R^{-1}(U) = (u_x, u_y, u_z) \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix}$$

$$= (u_x^2 + u_y^2 + u_z^2, 0, 0)$$

$$= (1, 0, 0)$$

$$= X$$

- Comment: Used for mapping between $XY$ plane and arbitrary plane (why is this useful!?)
Transformation of Objects

- Lemma: Affine image of line segment is line segment between image of its end points

- Proof:
  - Given affine transformation $A$ & line segment $S(t) = P_1t + P_2(1-t)$

  $$A(S(t)) = A(P_1t + P_2(1-t))$$
  $$= A(P_1t) + A(P_2(1-t))$$
  $$= tA(P_1) + (1-t)A(P_2)$$

- Is an affine image of a plane, a plane?
- Conclusion: polyhedron (polygon) is affinely transformed by transforming its vertices
Viewing Transformations

- Question: How can we view (draw) 3D objects on a 2D screen?
- Answer:
  - Project the transformed object along $Z$ axis onto $XY$ plane - and from there to screen (space)
  - Canonical projection:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 
\end{bmatrix}
$$

- In practice “ignore” the $z$ axis – simply use the $x$ and $y$ coordinates for screen coordinates.
Parallel Projection

- Projectors are all parallel
  - Orthographic: Projectors perpendicular to projection plane
  - Axonometric: Rotation + translation + orthographic projection
Parallel Projection

- Projectors are all parallel
  - Oblique projection: Projectors not necessarily perpendicular to projection plane

- When can the oblique projection be used?
What is Wrong in this Korean Picture?
Perspective Projection

- Viewing is from **point at a finite distance**

- Without loss of generality:
  - Viewpoint at origin
  - Viewing plane is \( z = d \)

- Given \( P = (x, y, z) \), triangle similarity gives:

\[
\frac{x}{z} = \frac{x_p}{d} \quad \text{and} \quad \frac{y}{z} = \frac{y_p}{d} \quad \Rightarrow \quad x_p = \frac{x}{z/d} \quad \text{and} \quad y_p = \frac{y}{z/d}
\]
Perspective Projection (cont’d)

- In matrix notation with homogeneous coordinates:

\[
P(x, y, z, l) = (x, y, z, l) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/d \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z/d \end{bmatrix}.
\]

- In Euclidean coordinates:

\[
\begin{bmatrix} x \\ z/d \\ y \\ z/d \end{bmatrix} = \begin{bmatrix} x \\ z/d \\ y \\ z/d \end{bmatrix} = (x_p, y_p, d).
\]

- \(P\) not injective & singular: \(\text{det}(P) = 0\)
Perspective Projection (cont’d)

- What is (if any) the difference between:
  - Moving the projection plane
  - Moving the viewpoint (center of projection)? Aka Dolly.
Perspective Projection (cont’d)

Scale

Dolly
Examples of Perspective projection

Perspective could be defined with

*one, two or three* focal points:
Examples of Perspective projection

The same model in

Orthographic view  Perspective view

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Examples of Perspective projection

The same model with

Small perspective warp   Large perspective warp
More than Three Points’ Perspective!

One Point Perspective

Three Point Perspective

Two Point Perspective

Four Point Perspective

Five Point Perspective

Six Point Perspective
Perspective Warp

- $P$ not invertible – can not get depth (order) back.
- Idea:
  - Warp the viewing frustum (and the entire world)
  - Then use parallel projection

Question: Why do we require the depth information?

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Perspective Warp

Matrix formulation

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & d & 1 \\
0 & 0 & d - \alpha & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
= \begin{bmatrix}
x, y, (z - \alpha)d, z \\
\end{bmatrix}
\]

This matrix maps the \( x \) and \( y \) coordinates much like the perspective map in slide 34.
Perspective Warp

- What does $\alpha = 0$ mean? $\alpha = d$?
- What is the effect of changing $d$, $d > \alpha$?
- The perspective warp preserves relative depth (third coordinate). For what depth domain? Prove it!
- The perspective warp preserves planes (prove it!)
- Plane $Ax + By + Cz + D = 0$ is warped to (prove it!):
  $$A\alpha dx_p + B\alpha dy_p + D(\alpha - d)z_p + C\alpha d^2 + Dd^2 = 0$$
- What does warp do to other objects? Spheres?
Perspective Warp

We will show that the six faces of the view frustum are mapped to six faces of a box.

- Plane $z = d$ ($d \neq 0$), $A=B=0$, $C=1$, $D=-d$ and the plane is warped to $-d(\alpha - d)z_p + \alpha d^2 - d^3 = 0$ or $z_p = d$

- Plane $z = \alpha$ ($\alpha \neq 0$), $A=B=0$, $C=1$, $D=-\alpha$ and the plane is warped to $-\alpha(\alpha - d)z_p + \alpha d^2 - \alpha d^2 = 0$ or $z_p = 0$

- Plane $x = \pm z$, $A=1$, $B=0$, $C= \pm 1$, $D=0$ and the plane is warped to $\alpha dx_p \pm \alpha d^2 = 0$ or $x_p = \pm d$

- Plane $y = \pm z$, $A=0$, $B=1$, $C= \pm 1$, $D=0$ and the plane is warped to $\alpha dy_p \pm \alpha d^2 = 0$ or $y_p = \pm d$
What is the difference between

\[ O = M N o \ & \ O = N M o, \ N \text{ new transform} \]
Another Transformations Quiz

- What does each transformation preserve?

<table>
<thead>
<tr>
<th>Transformation</th>
<th>lines</th>
<th>parallel lines</th>
<th>distance</th>
<th>angles</th>
<th>normals</th>
<th>convexity</th>
<th>Conics (sphericity)</th>
</tr>
</thead>
<tbody>
<tr>
<td>scaling</td>
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</tr>
</tbody>
</table>

- A map that preserves angles is called conformal.
- A map that preserves distances is called an Isometry.
Everything is a matter of Perspective

Sidewalk Chalk Paintings

Julian Beever